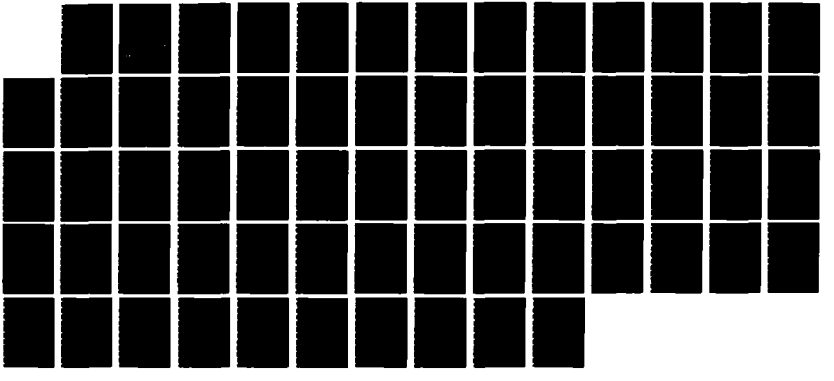
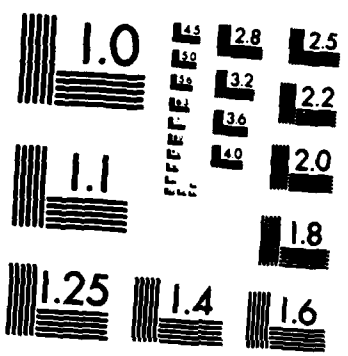
AD-A172 223 SIGNAL-TO-NOISE RATIO REQUIREMENTS FOR DETECTION OF MULTIPLE PULSES SUBJE... (U) NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CT NEW LONDON LAB... 1/1

UNCLASSIFIED A H NUTTALL ET AL. 02 JUN 86 NUSC-TR-7707 F/G 9/3 NL





(2)

NUSC Technical Report 7707
2 June 1986

# Signal-to-Noise Ratio Requirements for Detection of Multiple Pulses Subject to Partially Correlated Fading with Chi-Squared Statistics of Various Degrees of Freedom

Albert H. Nuttall
Edward S. Eby

Surface Ship Sonar Department



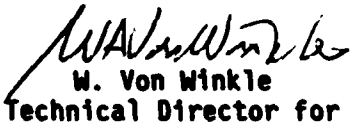
Naval Underwater Systems Center
Newport, Rhode Island / New London, Connecticut

## Preface

This research was conducted under NUSC Project No. J20024, "Surface Ship ASW Advanced Development," Principal Investigator Ira B. Cohen (Code 33142), Program Manager David M. Ashworth (Code 33A4), sponsored by Naval Sea Systems Command, Program Element 63553N, Subproject/Task S1704, Program Manager CDR Edward Graham IV (NAVSEA 63D). Also this research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator Dr. Albert H. Nuttall (Code 3314), sponsored by the NUSC In-House Independent Research Program, Program Manager Mr. W. R. Hunt, Director of Navy Laboratories (SPAWAR 05).

The Technical Reviewer for this report was Ira B. Cohen (Code 33142).

Reviewed and Approved: 2 June 1986



W. Von Winkle
Associate Technical Director for Technology

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

AD-A172 223

# REPORT DOCUMENTATION PAGE

| | |
|---|---|
| 1a. REPORT SECURITY CLASSIFICATION<br>UNCLASSIFIED | 1b. RESTRICTIVE MARKINGS |
| 2a. SECURITY CLASSIFICATION AUTHORITY | 3. DISTRIBUTION/AVAILABILITY OF REPORT<br>Approved for public release; distribution unlimited. |
| 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE | |
| 4. PERFORMING ORGANIZATION REPORT NUMBER(S)<br>TR 7707 | 5. MONITORING ORGANIZATION REPORT NUMBER(S) |
| 6a. NAME OF PERFORMING ORGANIZATION<br>Naval Underwater Systems Center<br>6b. OFFICE SYMBOL (If applicable) | 7a. NAME OF MONITORING ORGANIZATION |
| 6c. ADDRESS (City, State, and ZIP Code)<br>New London Laboratory<br>New London, CT 06320 | 7b. ADDRESS (City, State, and ZIP Code) |
| 8a. NAME OF FUNDING/SPONSORING ORGANIZATION<br>Naval Sea Systems Command<br>8b. OFFICE SYMBOL (If applicable) | 9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER |
| 8c. ADDRESS (City, State, and ZIP Code)<br>Department of the Navy<br>Washington, DC 20362 | 10. SOURCE OF FUNDING NUMBERS<br>PROGRAM ELEMENT NO. / PROJECT NO. J20024 & A75205 / TASK NO. / WORK UNIT ACCESSION NO. |

11 TITLE (Include Security Classification)
SIGNAL-TO-NOISE RATIO REQUIREMENTS FOR DETECTION OF MULTIPLE PULSES SUBJECT TO PARTIALLY CORRELATED FADING WITH CHI-SQUARED STATISTICS OF VARIOUS DEGREES OF FREEDOM

12 PERSONAL AUTHOR(S)
Albert H. Nuttall and Edward S. Eby

| 13a. TYPE OF REPORT | 13b. TIME COVERED FROM ____ TO ____ | 14. DATE OF REPORT (Year, Month, Day) | 15 PAGE COUNT |
|---|---|---|---|
| | | 2 June 1986 | |

16 SUPPLEMENTARY NOTATION
—

| 17 COSATI CODES | | | 18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) |
|---|---|---|---|
| FIELD | GROUP | SUB-GROUP | Signal-to-Noise Ratio; Chi-Square Fading; Multiple Pulses; Diversity Combination; Correlated Fading; Matched Filters |
| | | | |
| | | | |

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

The transmitted signal in a fading medium is composed of several pulses separated in time so as to achieve diversity and thereby combat deep fades and loss of signal. Receiver processing consists of matched filtering of each of the pulses, followed by summation of the squared envelopes of all the filter outputs. In addition to additive Gaussian background noise, the signal is subject to slow medium fading which has a chi-squared first-order distribution and which may be correlated from pulse to pulse to an arbitrary degree.

The false alarm and detection probabilities of this system are derived in various series expansions which are amenable to efficient computer evaluation. Programs are presented and exercised for various combinations of signal-to-noise ratio, number of pulses, degree of correlated fading, and (noninteger) number of degrees of freedom of the chi-squared fading. Required input signal-to-noise ratios for several false alarm

| 20 DISTRIBUTION/AVAILABILITY OF ABSTRACT | 21 ABSTRACT SECURITY CLASSIFICATION |
|---|---|
| ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS | UNCLASSIFIED |

| 22a NAME OF RESPONSIBLE INDIVIDUAL | 22b TELEPHONE (Include Area Code) | 22c OFFICE SYMBOL |
|---|---|---|
| Albert H. Nuttall | (203) 440-4618 | Code 3314 |

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted; All other editions are obsolete

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

18. (Cont'd.)

Detection Probabilities
Slow Fading
Additive Gaussian Noise
Effective Number of Independent Scalings
Characteristic Function
Exceedence Distribution

19. (Cont'd.)

and detection probabilities are computed and plotted for cases of the fading normalized correlation coefficient ranging from 0 to 1; results for a nonfading medium are superposed for easy comparison.

As special cases of the above formulation, we have independent Rayleigh amplitude fading, as well as completely dependent Rayleigh amplitude fading.

# TABLE OF CONTENTS

# LIST OF ILLUSTRATIONS

# LIST OF SYMBOLS

| | |
|---|---|
| $t$ | Time |
| $L$ | Duration of each signal pulse |
| $f_o$ | Carrier frequency |
| $p(t)$ | Pulse waveform, (1) |
| $\Theta_o$ | Phase of pulse carrier |
| $A$ | Amplitude of transmitted signal pulse, (2) |
| $K$ | Number of transmitted pulses |
| $t_k$ | Transmission time of k-th pulse |
| $r_k$ | Amplitude scaling due to propagation, (4) |
| $D$ | Bulk delay in propagation |
| $\tilde{p}$ | Randomly phase-shifted signal pulse |
| $\Theta_k$ | Random phase shift of k-th pulse |
| Overbar | Ensemble average |
| $\overline{E_1}$ | Average received signal energy per pulse, (6) |
| $n(t)$ | Additive zero-mean white Gaussian noise |
| $N_d$ | Double-sided noise spectral density level |
| $N_o$ | Single-sided noise spectral density level, $N_o = 2N_d$ |
| $w(t)$ | Received waveform, (7) |
| $\gamma$ | System output; decision variable |
| $f_\gamma$ | Characteristic function of $\gamma$, (8) |
| $\xi$ | Argument of characteristic function |
| $\sigma_n^2$ | Noise variance, (9) |
| $\sigma_T^2$ | Total variance, (9) |

## LIST OF SYMBOLS (Cont'd)

| | |
|---|---|
| $N$ | Parameter of characteristic function, (10) |
| $R$ | Signal-to-noise ratio parameter, (9) |
| $K_e$ | Effective number of independent fades, (10) |
| $\rho_{k\ell}$ | Covariance coefficient of fading, (10),(15) |
| $u$ | Argument of probability density function |
| $p_r$ | Probability density function of $r_k$ |
| $m$ | Shape parameter in density of $r_k$, (11) |
| $\alpha$ | Strength parameter in density of $r_k$, (11) |
| $q_k$ | Power scaling due to propagation, $r_k^2$, (12) |
| $\sigma_q$ | Standard deviation of $q_k$ |
| $Q_\gamma$ | Exceedance distribution function of $\gamma$, (16) |
| $P_D$ | Probability of detection, (17) |
| $\Lambda$ | Normalized threshold, (18) |
| $e_j(x)$ | Partial exponential, (19) |
| $P_F$ | Probability of false alarm, (20) |
| $S$ | Sum of power scalings, (22) |
| $E_k$ | Received signal energy on k-th pulse for no fading, (23) |
| $E_T$ | Total received signal energy for no fading, (24) |
| $Q_M(a,b)$ | $Q_M$-function, (28) |
| PDF | Probability density function |
| SNR | Signal-to-noise ratio |
| NF | No fading |

# INTRODUCTION

In many situations, the operating environment of an active pulsed system (radar, communications, or sonar) limits the number of pulses transmitted. Moreover the pulses transmitted through a medium, reflected, and returned can be mathematically modeled as received pulses with pulse-to-pulse amplitude variation (fading). An air traffic control radar in a mountainous region operating during severe weather might be subject to such limitations. High frequency radio communication at long ranges (skip distances), when ionospheric conditions are changing because of normal diurnal processes or during solar storms, provide a second class of examples. Replacement of these electromagnetic systems by their acoustic analogues provides the further instance of a sonar operating among islands or at convergence zone ranges (underwater skip distances). The signal-to-noise ratio at the system input could be randomly reduced to a value so low as to preclude acceptable detection performance from processing each signal pulse independently.

Detection performance of a processing system can be mathematically characterized by the parameters $P_D$, probability-of-detection, and $P_F$, probability-of-false-alarm. When several pulses are transmitted in a pulse train burst, and the returned echoes are processed as a group, we wish to determine the minimum required input signal-to-noise ratio, as a function of the number of pulses, K, in the burst, necessary to achieve the preassigned acceptable level of performance, $P_D$,$P_F$. Similarly, if only a fixed total

energy is available for transmission, we want to select the "best" distribution of the available energy into K equal duration pulses to ensure acceptable system performance. Stated differently, what is the optimal number of baskets (pulses) for our eggs (energy)?

For our purposes here, a transmitted burst consists of K identical, nonoverlapping (in time), sinusoidal pulses. The pulse train received is mathematically modeled as time-delayed, randomly phase-shifted, versions of the transmitted pulses with random pulse-to-pulse amplitude variations of known first-order probability density. This received signal is further corrupted by additive Gaussian white noise. The fading is assumed to be unnoticeably slow during a pulse, so that the only fades it is necessary to model are from pulse-to-pulse, which can be correlated to an arbitrary degree.

One possible detection tactic might be to transmit the smallest energy signal just necessary to achieve the required average detection performance, and accomplish it with a single pulse. Because of a variety of limitations, either physical or financial, this tactic may not achieve the desired performance. Multipulse bursts might work in cases where the available signal-to-noise ratio is low. Here we consider in what sense this intuition is correct for one class of correlated fading, and we compare these results with system performance in a nonfading medium.

In the next section, the characteristics of the transmitted and received signals, the corrupting additive noise, and the receiver processing are

mathematically modeled. This provides notation and scenario for the analysis of the hypothesized receiver, depicted in figure 1. The class of power scaling

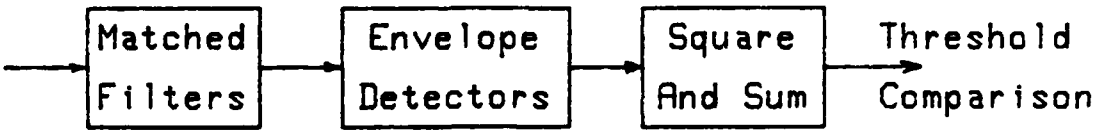



Figure 1. Receiver Block Diagram

fading variates considered have chi-square densities so, in a sense, the received envelope amplitude scalings can be thought of as the square roots of chi-square variates. The possibility that several pulses in a pulse train are correlated is covered as well. Numerical examples for several combinations of $P_D$ and $P_F$, for some representative received scaling probability density functions and pulse-to-pulse correlations, are presented and discussed.

Two appendices provide mathematical details supporting the main analysis by deriving the characteristic function and exceedance distribution of the system output; a third appendix lists computer programs for the computation of required input signal-to-noise ratios for fading and nonfading received signals.

# PROBLEM DEFINITION

## Signal Characteristics

The detection problem of interest is described in this section. A high-frequency pulse of length L seconds and carrier $f_0$ Hz is utilized as the basic signal waveform component:

$$p(t) = \cos(2\pi f_0 t + \Theta_0) \quad \text{for } 0 < t < L \ . \tag{1}$$

We have $Lf_0 >> 1$; that is, each pulse contains many cycles of the carrier. The actual transmitted signal waveform is

$$A \sum_{k=1}^{K} p(t - t_k) \ , \tag{2}$$

where A is the transmitted signal peak amplitude, K is the number of pulses, and $\{t_k\}$ are the transmission times which are arbitrary except that

$$t_{k+1} - t_k > L \ ; \quad \text{i.e., no overlap} \ . \tag{3}$$

The received signal is

$$A \sum_{k=1}^{K} r_k \, \tilde{p}(t - t_k - D) \ , \tag{4}$$

where $\{r_k\}$ are amplitude scalings imposed by propagation and attenuation through the medium. Bulk delay D is common to all pulses, and $\tilde{p}$ denotes a randomly phase-shifted version of (1) by independent uniformly distributed phase shifts $\{\theta_k\}$ for $1 \leq k \leq K$.

The average received signal power during a single pulse is

$$A^2 \overline{r_k^2} \overline{\cos^2(2\pi f_o(t - t_k - D) + \theta_o + \theta_k)} = \frac{1}{2} A^2 \overline{r^2} , \qquad (5)$$

and the average received signal energy per pulse is then

$$\overline{E_1} = \frac{1}{2} A^2 \overline{r^2} L . \qquad (6)$$

This is a very important parameter regarding the received signal; it is independent of pulse number k under the assumption that $\overline{r_k^2} = \overline{r^2}$, that is, stationary fading.

Noise Characteristics

There is also assumed present at the receiver input, additive zero-mean white Gaussian noise n(t) with a double-sided spectrum level $N_d$ watts/Hz, or equivalently, a single-sided spectrum level $N_o = 2N_d$. The total received waveform is therefore,

$$w(t) = A \sum_{k=1}^{K} r_k \tilde{p}(t - t_k - D) + n(t) . \qquad (7)$$

Receiver Processing

The bulk delay D, as well as the time separations $\{t_k\}$ between pulses, are assumed known at the receiver; otherwise, a search over time delay D is required, in addition to a possible Doppler search, in general. The receiver consists of K matched filters, one for each of the signal pulses in (4); these filter outputs are envelope detected and sampled at the appropriate instants corresponding to the peak signal outputs. Finally, these envelope samples are squared and summed over the total of K pulses. This sum is then compared with a threshold for decision about signal presence or absence; see figure 1.

The analytic problem of interest is to determine the probability that the decision variable above exceeds a threshold. From this quantity, we can determine the detection and false alarm probabilities as functions of all the various parameters presented in (1) - (7). Then for specified detection and false alarm probabilities, the required input signal-to-noise ratio, $\overline{E_1}/N_0$, can be determined numerically.

## ANALYTIC RESULTS

### Characteristic Function of Decision Variable

The derivations of the statistics of the decision variable $\gamma$ are conducted in appendices A and B. Specifically, the characteristic function of $\gamma$ is given in (A-31) as

$$f_\gamma(\xi) = \left(1 - i\xi 2\sigma_n^2\right)^{N-K} \left(1 - i\xi 2\sigma_T^2\right)^{-N} , \tag{8}$$

where

$$\sigma_n^2 = \frac{1}{4} N_o L , \quad \sigma_T^2 = \sigma_n^2(1 + R) , \quad R = \frac{\overline{E_1}}{N_o} \frac{K}{N} ,$$

$$\frac{\overline{E_1}}{N_o} = \frac{\text{average received signal energy per pulse}}{\text{single-sided received noise spectral density level}} , \tag{9}$$

and

$$N = m K_e , \quad K_e = \frac{K^2}{\sum_{k,\ell=1}^{K} \rho_{k\ell}} . \tag{10}$$

Here, m is a parameter in the first order probability density function of amplitude scalings $\{r_k\}$, namely,

$$p_r(u) = \frac{2 u^{2m-1} \exp(-u^2/\alpha)}{\Gamma(m)\, \alpha^m} \quad \text{for } u > 0 \quad (m > 0) . \tag{11}$$

To explain the other parameters in (10), we use the power scaling variable

$$q_k = r_k^2 \tag{12}$$

for the k-th pulse in (4) and (7). From (11), the probability density function of $\{q_k\}$ is

$$p_q(u) = \frac{u^{m-1}\exp(-u/\alpha)}{\Gamma(m)\,\alpha^m} \quad \text{for } u > 0 \quad (m > 0) . \tag{13}$$

This is recognized as a chi-squared variate of 2m degrees of freedom; however, 2m need not be an integer. It is readily verified that

$$\frac{\text{Std Dev}(q)}{\text{Mean}(q)} = \frac{\sigma_q}{\bar{q}} = \frac{1}{\sqrt{m}} . \tag{14}$$

Thus as $m \to \infty$, the probability density function of power scaling $q_k$ narrows about its mean value,* while if $m \to 0$, it develops a large cusp at the origin and a significant spread about the mean.

The normalized covariance coefficient appearing in (10) is that of the power scaling variates:

$$\rho_{k\ell} = \frac{1}{\sigma_q^2}\,\overline{(q_k - \bar{q})(q_\ell - \bar{q})} . \tag{15}$$

It measures the degree of correlation between the fading imposed on the signal pulses, i.e., $\{r_k\}$ in (7).

---

*This corresponds to a nonfading medium.

The quantity $K_e$ in (10) can be interpreted as an effective number of independent scalings in (4) and (7). For example, if $\rho_{k\ell} = 0$ for $k \neq \ell$, then $K_e = K$; while if $\rho_{k\ell} = 1$ for all $k,\ell$, then $K_e = 1$. Both of these situations agree with physical intuition. No specific time separations $\{t_k\}$, in (4) and (7), need be assumed for (10) to apply; some of the pulses can be close together, while others can be widely separated, subject of course to limitation (3). The only way that the statistical dependencies of the power scalings $\{q_k\}$ enter the characteristic function of $\gamma$ is through the double summation of covariance coefficients in (10).

However, it must be remarked that the result in (8) is only an approximation, developed in appendix A. An exact analysis of the characteristic function of output $\gamma$ is the subject of a future study.

Exceedance Distribution Function of Decision Variable

Three alternative forms for the exceedance distribution function of $\gamma$,

$$Q_\gamma(u) = \text{Prob}(\gamma > u) \ , \tag{16}$$

are developed in appendix B, namely, (B-9), (B-11), and (B-13), each having different merits, as discussed there. The one we have used for our numerical calculations is the last one, and is, in fact, the detection probability when signal is present:

$$P_D = 1 - \frac{\exp(-\Lambda)}{(1+R)^N} \sum_{n=0}^{\infty} \frac{(N)_n}{n!} \left(\frac{R}{1+R}\right)^n \left[\exp(\Lambda) - e_{K-1+n}(\Lambda)\right] . \tag{17}$$

Here

$$\Lambda = \frac{u}{2\sigma_n^2} \tag{18}$$

is a normalized threshold, and

$$e_j(x) = \sum_{n=0}^{j} x^n/n! \tag{19}$$

is the partial exponential [1; 6.5.11]. The other two parameters appearing in $P_D$, namely, R and N, have already been explained in (9) and (10).

The false alarm probability corresponding to (17) is obtained by setting R = 0:

$$P_F = \exp(-\Lambda)\ e_{K-1}(\Lambda) . \tag{20}$$

Thus, the performance of the diversity combining processor is governed by the fundamental pair of equations, (17) and (20). For a specified number of pulses K and false alarm probability $P_F$, (20) can be solved for threshold $\Lambda$. These values of K and $\Lambda$ are then employed in (17) to evaluate detection probability $P_D$ for any specified N and $\overline{E}_1/N_o$. A program for this procedure is given in appendix C.

Detection Probability for Nonfading Medium

For comparison purposes, the detection probability for the processor of figure 1 in the presence of a deterministic, i.e., nonfading, medium is also presented. This result is most easily obtained from the conditional characteristic function for the system output, as presented in (A-13) and (A-14):

$$f_Y^{(c)}(\xi) = \left(1 - i\xi 2\sigma_n^2\right)^{-K} \exp\left[\frac{i\xi}{1 - i\xi 2\sigma_n^2}\left(\frac{AL}{2}\right)^2 S\right], \qquad (21)$$

where

$$S = \sum_{k=1}^{K} q_k \qquad (22)$$

is the sum of the power scalings on all K pulses. Since there is no fading, (21) is directly the characteristic function of the decision variable.

A more convenient form for (21) is available when we observe from (4) and (1) that the received signal energy on the k-th pulse is the nonrandom quantity

$$E_k = A^2 r_k^2 \frac{1}{2} L = \frac{1}{2} A^2 L q_k . \qquad (23)$$

Thus, the total received signal energy is, using (22),

$$E_T = \sum_{k=1}^{K} E_k = \frac{1}{2} A^2 L S . \qquad (24)$$

Coupled with the expression for $\sigma_n^2$ given in (9), we obtain for (21), the characteristic function,

$$f_\gamma^{(c)}(\xi) = \left(1 - i\xi 2\sigma_n^2\right)^{-K} \exp\left[\frac{i\xi 2\sigma_n^2}{1 - i\xi 2\sigma_n^2} \frac{E_T}{N_o}\right], \tag{25}$$

in terms of the parameter

$$\frac{E_T}{N_o} = \frac{\text{total received signal energy over K pulses}}{\text{single-sided received noise spectral density level}} . \tag{26}$$

The exceedance distribution function corresponding to (25) is available from [2; (50) and (51)] as

$$P_D = Q_K\left(\left(\frac{2E_T}{N_o}\right)^{1/2}, \left(2\lambda\right)^{1/2}\right), \tag{27}$$

where we used (18) and the $Q_M$-function defined as

$$Q_M(a,b) = \int_b^\infty dx\, x \left(\frac{x}{a}\right)^{M-1} \exp\left(-\frac{x^2 + a^2}{2}\right) I_{M-1}(ax) . \tag{28}$$

The false alarm probability corresponding to (27) is obtained by setting $E_T = 0$, and is identically (20), as it must be, since the background noise is independent of any signal fading characteristics.

When we plot these latter results for a nonfading medium, and compare them with the earlier results in (17) for the fading medium of interest, we replace $E_T$ by $K\,\overline{E_1}$. This arbitrary but reasonable assignment is necessary in order to superpose the two types of results on one plot. However, strictly speaking, the parameter $\overline{E_1}/N_o$ appearing in the following plots should be interpreted as $E_T/(KN_o)$ in the case of no fading.

## NUMERICAL RESULTS

The probability density functions (PDF) for five values of the scaling parameter m in (11) and (13), representing five distinct possible fading behaviors, are plotted in figure 2. All densities of the class considered vanish for negative argument. So these behaviors are, for the amplitude variate: (1) an impulse representing the nonfading case, which is also the limiting case of the class as $m \to \infty$; (2) continuous with continuous derivative at the origin (m = 2); (3) continuous with discontinuous derivative at the origin (m = 1, Rayleigh amplitude density); (4) a finite discontinuity at the origin (m = 1/2, a one-sided or full-wave rectified zero-mean Gaussian density); and (5) an infinite cusp at the origin (m = 1/4). The amplitude scaling r, with the probability density function of (11), has second moment

$$\overline{r^2} = m\alpha . \tag{29}$$

The normalized amplitude scaling

$$\hat{r} = r \Big/ \left(\overline{r^2}\right)^{1/2} , \tag{30}$$

normalized by the root-mean-square (rms) value of the amplitude scaling (not the standard deviation $\sigma_r$), has probability density function

$$p_{\hat{r}}(u) = \frac{2\, m^m\, u^{2m-1}\, \exp(-mu^2)}{\Gamma(m)} \quad \text{for } u > 0 . \tag{31}$$

For the five values of m ($=\infty$, 2, 1, 1/2, 1/4), the probability density functions for this rms-normalized amplitude scaling are shown in figure 2A.

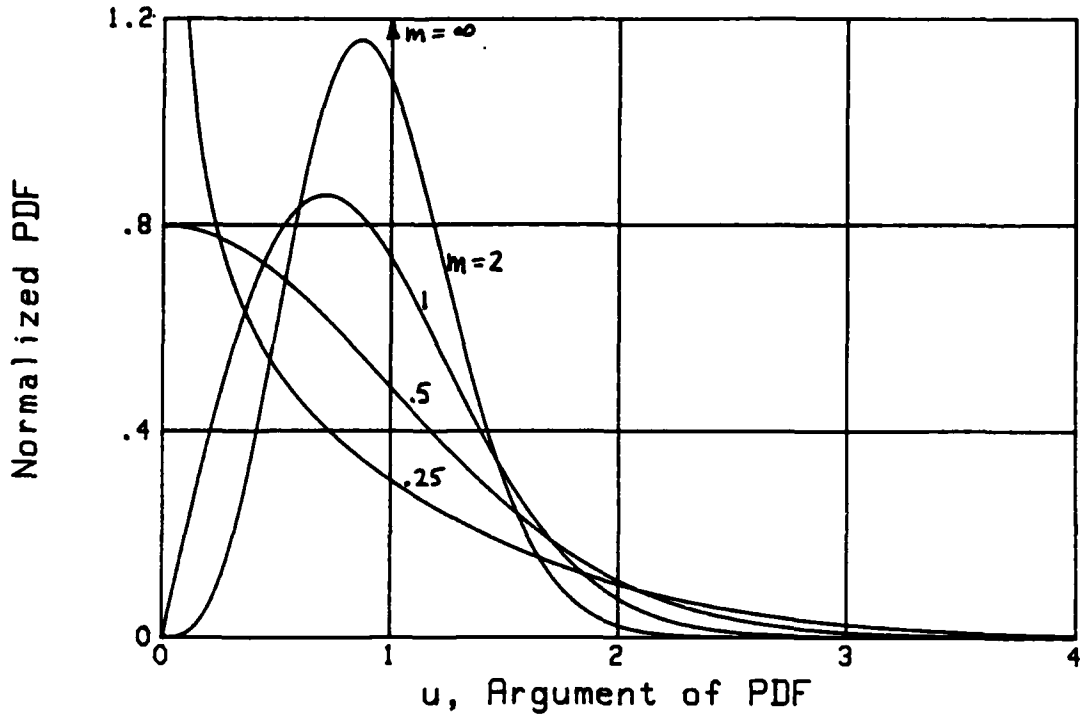



Figure 2A. PDF of Normalized Amplitude Scaling

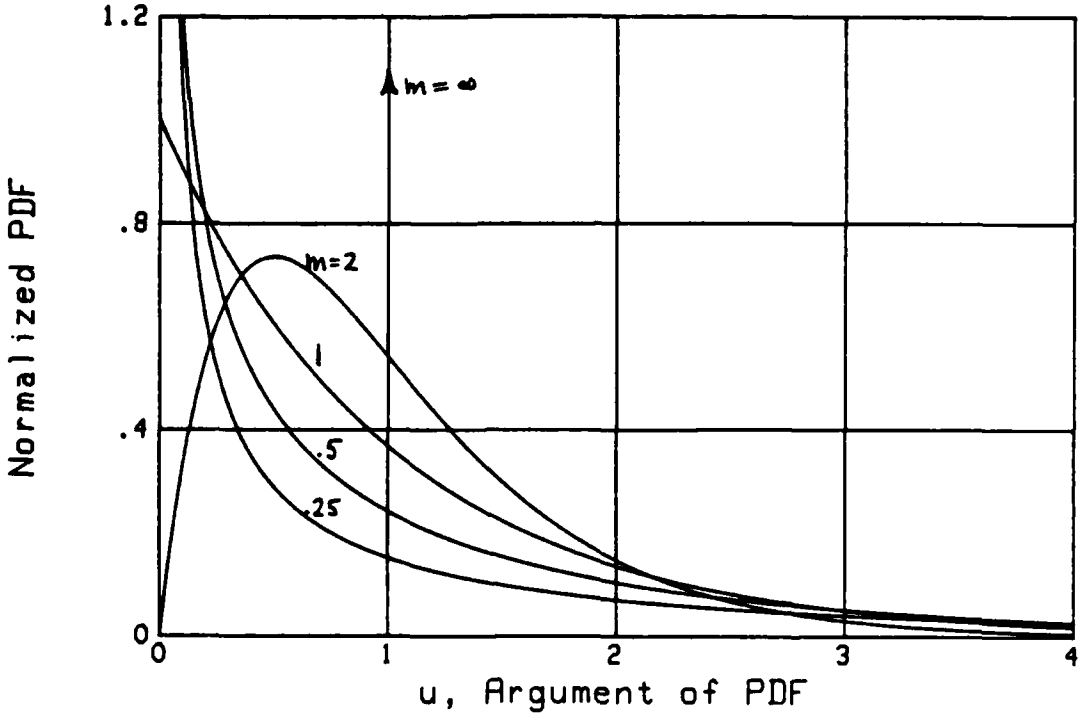



Figure 2B. PDF of Normalized Power Scaling

The corresponding probability density functions for power scaling q are chi-squared variates with 2m degrees of freedom, as given in (13). The mean of variate q is

$$\bar{q} = m\alpha . \tag{32}$$

The normalized power scaling

$$\tilde{q} = q/\bar{q} , \tag{33}$$

normalized by the mean power, has probability density function

$$p_{\tilde{q}}(u) = \frac{m^m u^{m-1} \exp(-mu)}{\Gamma(m)} \quad \text{for } u > 0 . \tag{34}$$

The normalized power scaling probability density functions are plotted in figure 2B for the same five values of m.

For each finite value of m above, and for each of the four combinations of system performance parameters $P_D$ = .5 or .9, and $P_F$ = 1E-6 or 1E-8, the required minimum system input signal-to-noise ratio (SNR) to achieve this performance, as a function of the number of pulses K in a pulse train, has been computed using the computer programs listed in appendix C which implement the theoretical analysis above. Although this analysis and these programs can handle general pulse-to-pulse normalized covariance coefficients $\{\rho_{k\ell}\}$, the figures have been computed for the special case of exponential correlation

$$\rho_{k\ell} = \rho^{|k-\ell|} , \quad -1 \leq \rho \leq 1 . \tag{35}$$

Each of the graphs has two sets of curves: the solid curves are for the required average received signal energy per pulse ($\overline{E}_1/N_0$) in dB, as a function of the number of pulses K, and the dotted curves are for the total average received signal energy ($\overline{E}_T/N_0$) in dB. For ease of comparison, each set of curves on each graph includes its nonfading (NF) or $m \to \infty$ counterpart. In every case computed, the solid curves show that the energy per pulse, or input signal-to-noise ratio required for acceptable system performance, decreases as the number of pulses increases. This matches the intuitive argument that using more pulses implies we are using more total energy, so we should be able to maintain the same level of performance with less energy per pulse. The intuitive conclusion is correct, but scanning the dotted sets of curves (representing total energy in the pulse train) shows this intuitive argument is only part of the story. For more-or-less uncorrelated adjacent pulse fading and for the smaller scaling parameters ($m \leq 1$), the total input signal energy required initially decreases as we increase the number of pulses in a pulse train. For a large number of pulses, the total energy will eventually increase with K, but for a small number of pulses, a short burst may be more efficient. For example, with $P_D = .5$, $P_F = 1E\text{-}8$, and a one-sided Gaussian amplitude probability density function, $m = 1/2$, figure 13 shows that the total energy required reaches a minimum of 14.4 dB for $\rho = 0$ with 3 pulses per burst; for $\rho = .5$, the minimum of 15.2 dB occurs with 5 pulses. In all cases, the $\rho = .5$ curve is closer to the $\rho = 0$ curve than to the $\rho = 1$ curve, and the high correlation $\rho = 1$ curve always increases. Thus, for moderate values of correlation, we can expect system performance to be robust and system performance will be severely degraded only for highly correlated fading.

As the exponential covariance coefficient $\rho$ in (35) decreases toward -1, the performance curves for even K approach those corresponding to the nonfading case. This is easily seen since the sum of two successive power scalings in the sum S in (22) and (A-14) then tends to $2\bar{q}$, twice the mean of q, which is a nonrandom quantity. Equivalently, we can say the random fluctuations in the power scalings $\{q_k\}$ have canceled each other. For K odd, however, there is always one unpaired random fluctuation, even for $\rho = -1$; this means that for K odd and $\rho = -1$, system performance is slightly poorer than for the nonfading situation. As K increases through odd values, this discrepancy decreases to insignificance. Even for K = 3, the difference is small. This has been confirmed analytically and verified by numerical calculation for values of $\rho$ very near -1, and for several K; these results have been omitted from the figures which include only nonnegative values of $\rho$.

A separate issue is whether values of $\rho$ near $\rho = -1$ can be realized. That is, if the first-order probability density function of the nonnegative power scaling variate q is not symmetric about its mean $\bar{q}$, then arbitrarily large negative values of $\rho$ are not allowed. For example, for $\rho_{k\ell}$ to reach -1, we would need $(q_k - \bar{q}) = -(q_\ell - \bar{q})$ with probability one. However, for the class of power scalings treated here, this is clearly impossible: large positive values of $q_k - \bar{q}$ occur with nonzero probability, for which the corresponding negative values $-(q_k - \bar{q})$ occur with probability zero. Thus, for most probability density functions, and in particular for the class assumed here, the $\rho = -1$ results would represent an unreachable bound to

system performance. This is an illustration of the fact that the first-order probability density function and the correlation (or spectrum) of a random process cannot be independently specified in general. Complete first-order information imposes some restrictions on allowable second-order statistics.

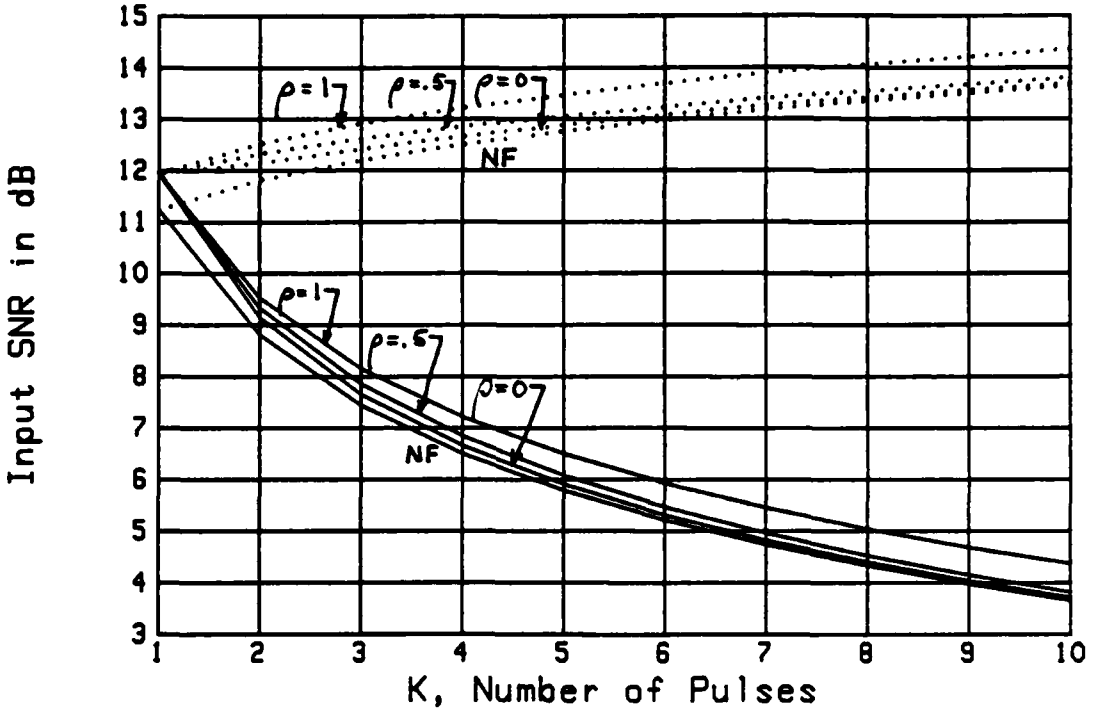



Figure 3. $P_F$ = 1E-6, $P_D$ = .5, m = 2

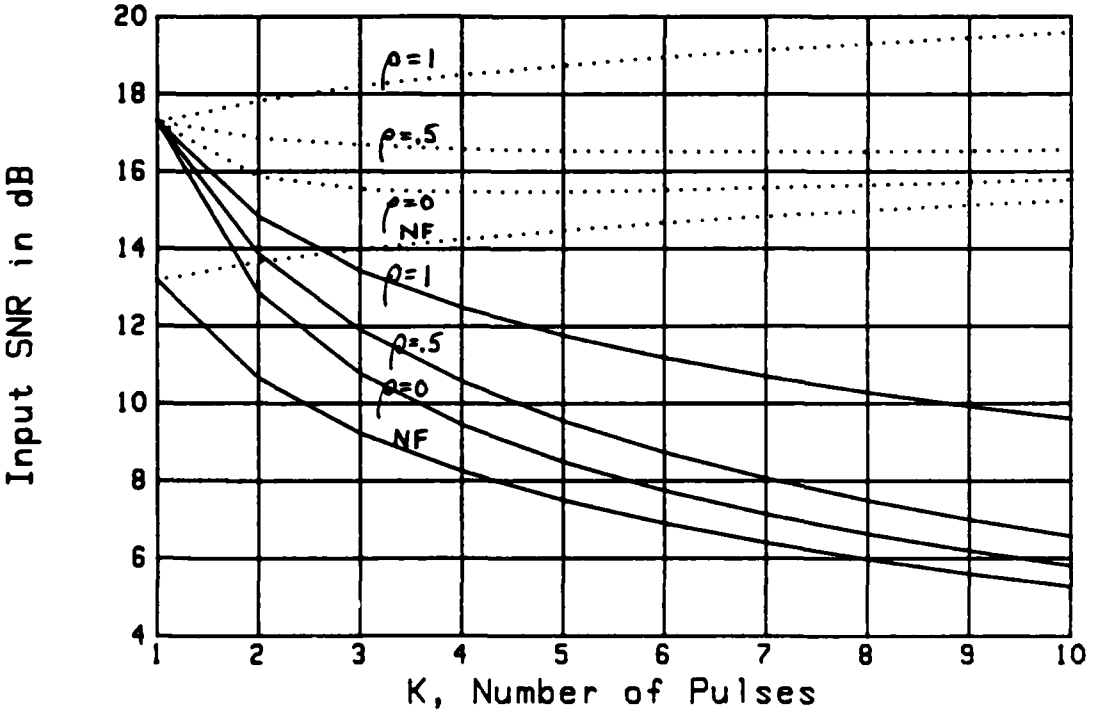



Figure 4. $P_F$ = 1E-6, $P_D$ = .9, m = 2

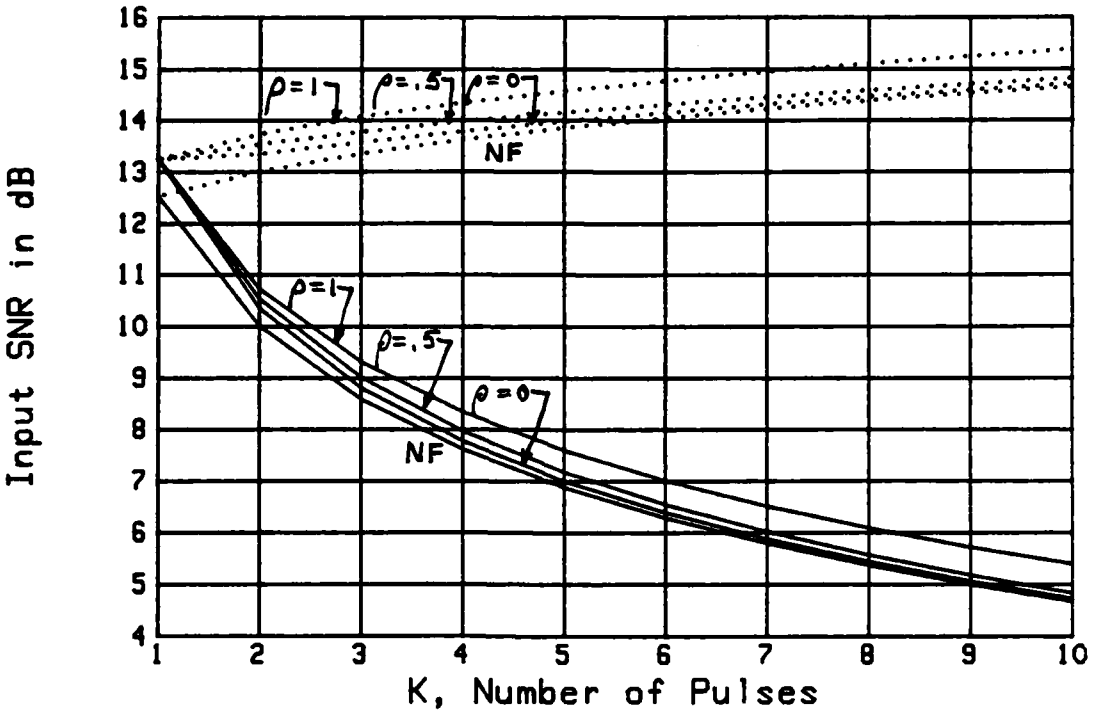



Figure 5. $P_F$ = 1E-8, $P_D$ = .5, m = 2

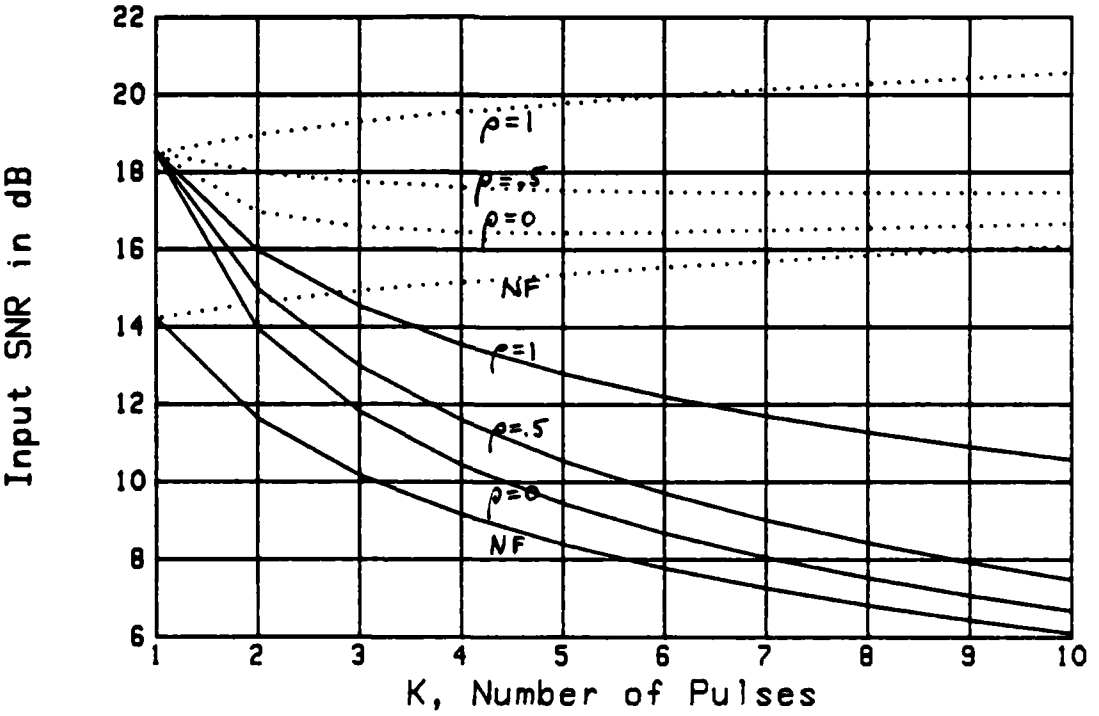



Figure 6. $P_F$ = 1E-8, $P_D$ = .9, m = 2

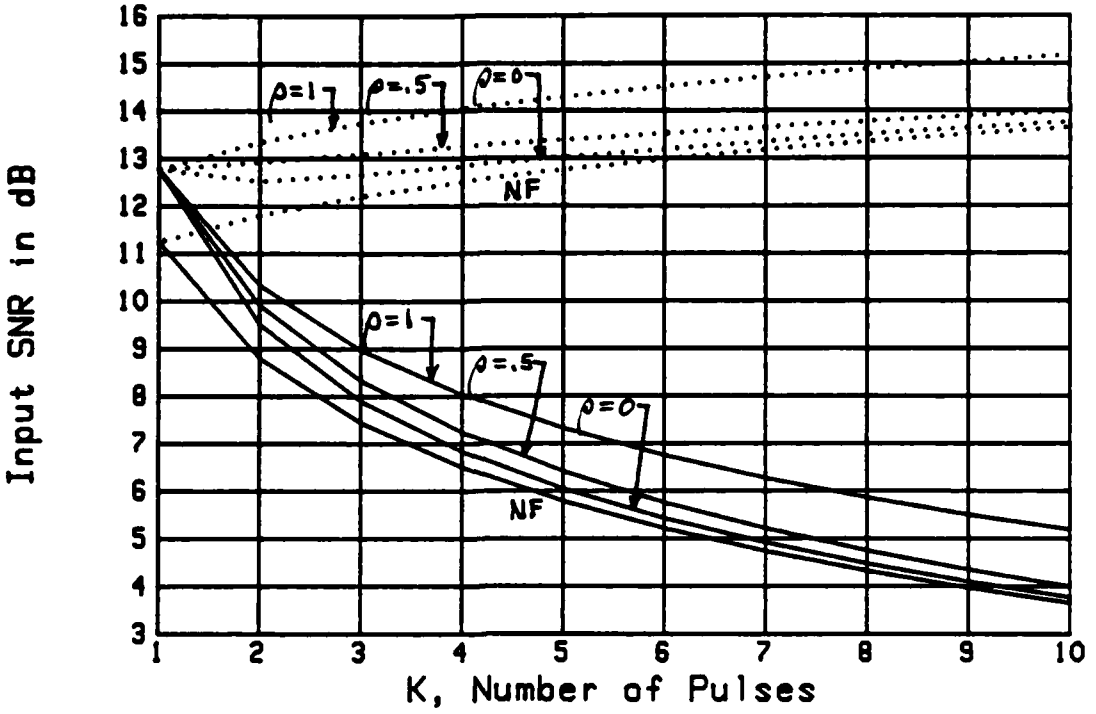



Figure 7. $P_F$ = 1E-6, $P_D$ = .5, m = 1

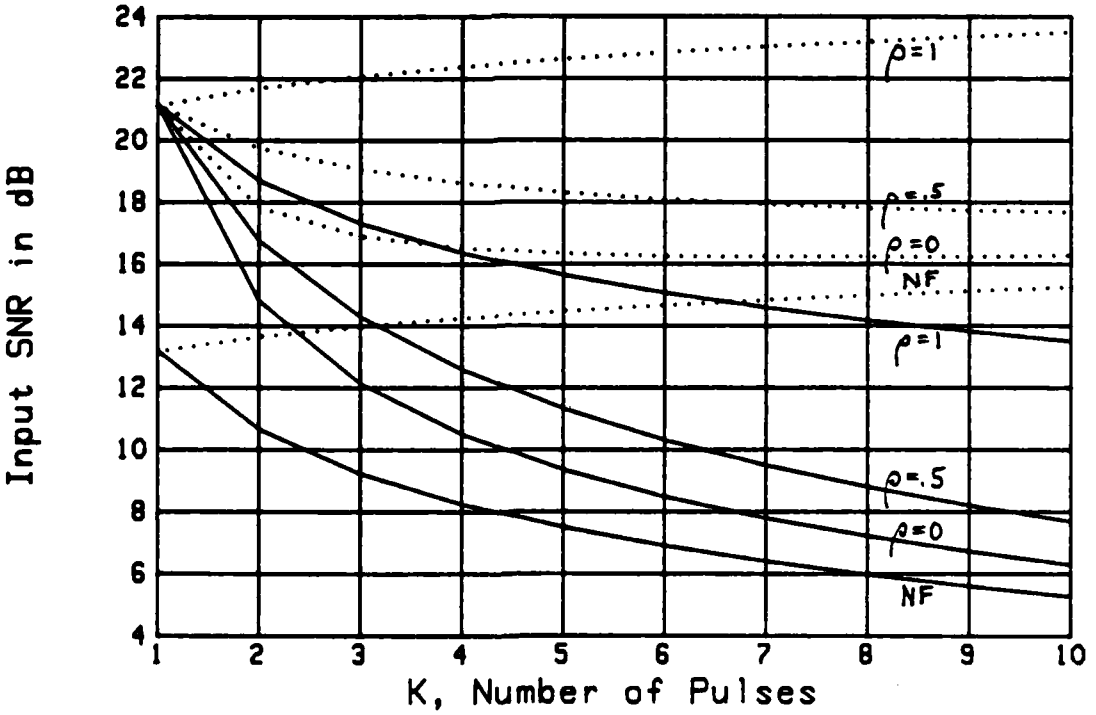



Figure 8. $P_F$ = 1E-6, $P_D$ = .9, m = 1

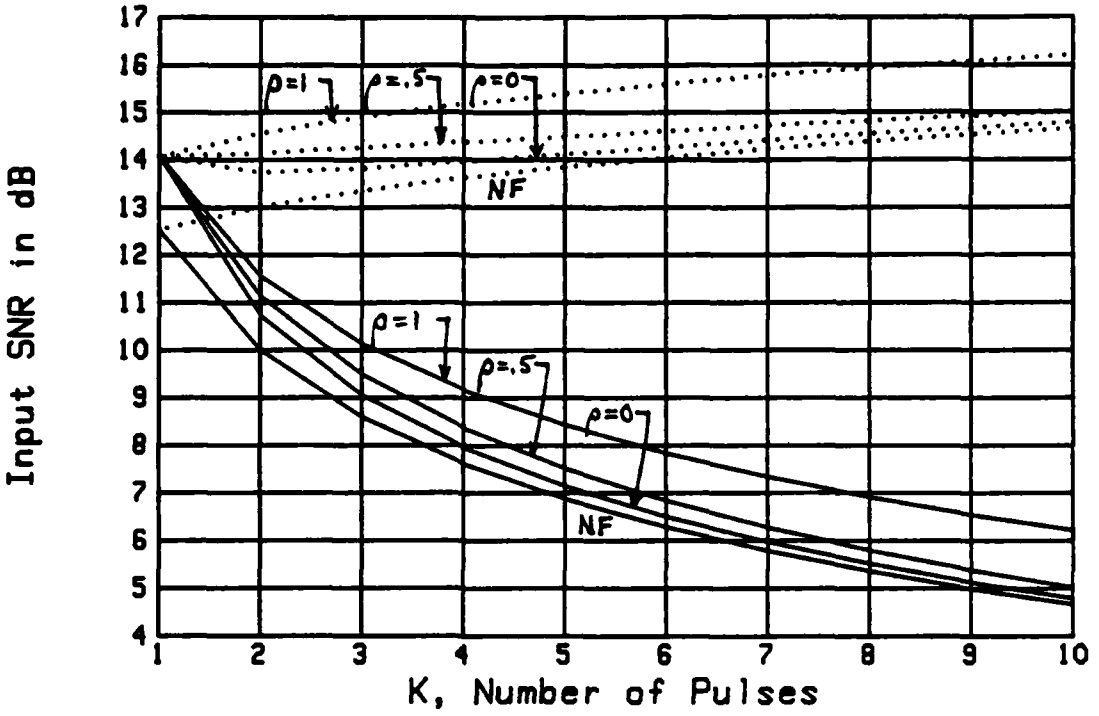



Figure 9. $P_F$ = 1E-8, $P_D$ = .5, m = 1

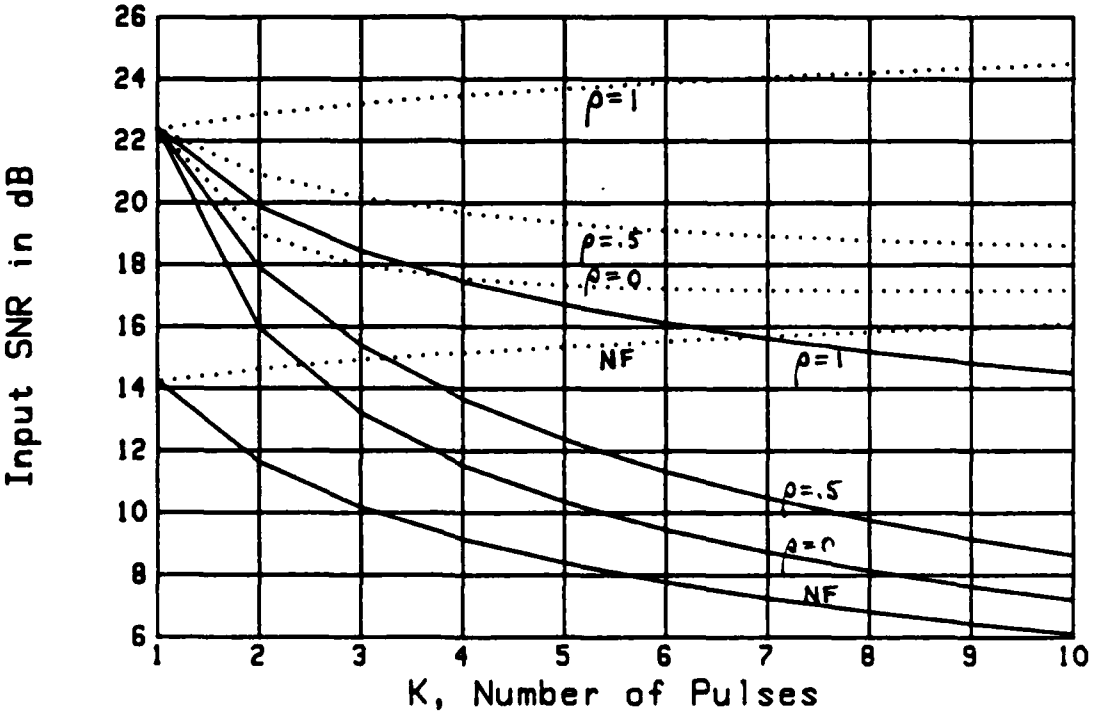



Figure 10. $P_F$ = 1E-8, $P_D$ = .9, m = 1

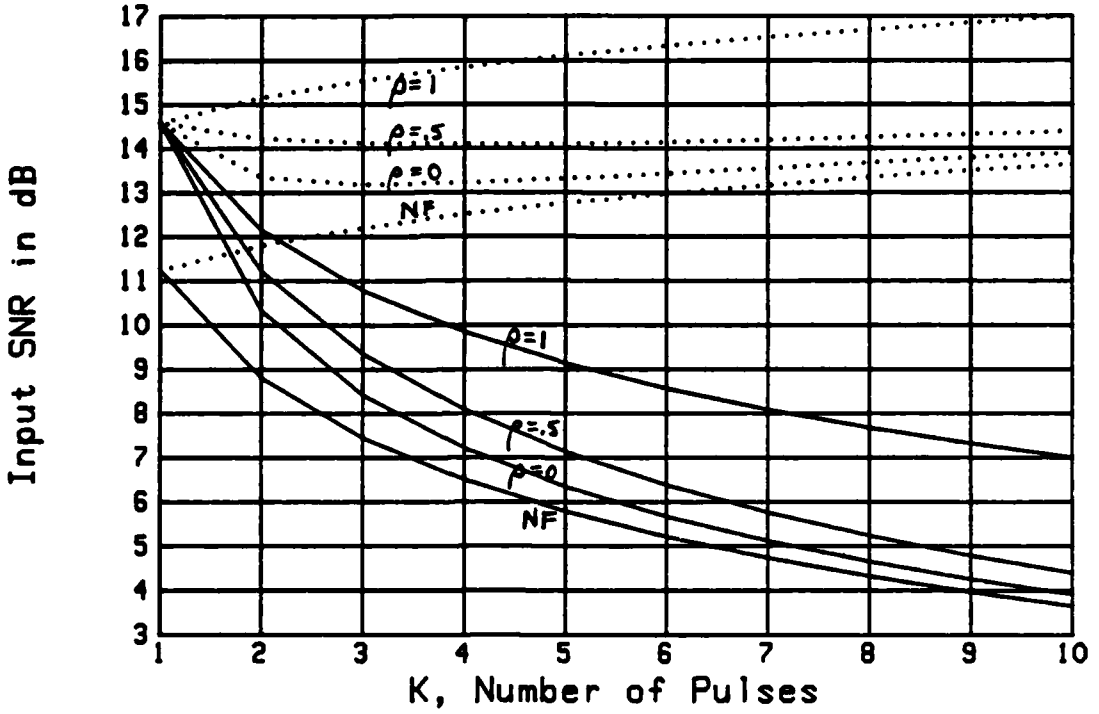



Figure 11. $P_F$ = 1E-6, $P_D$ = .5, m = .5

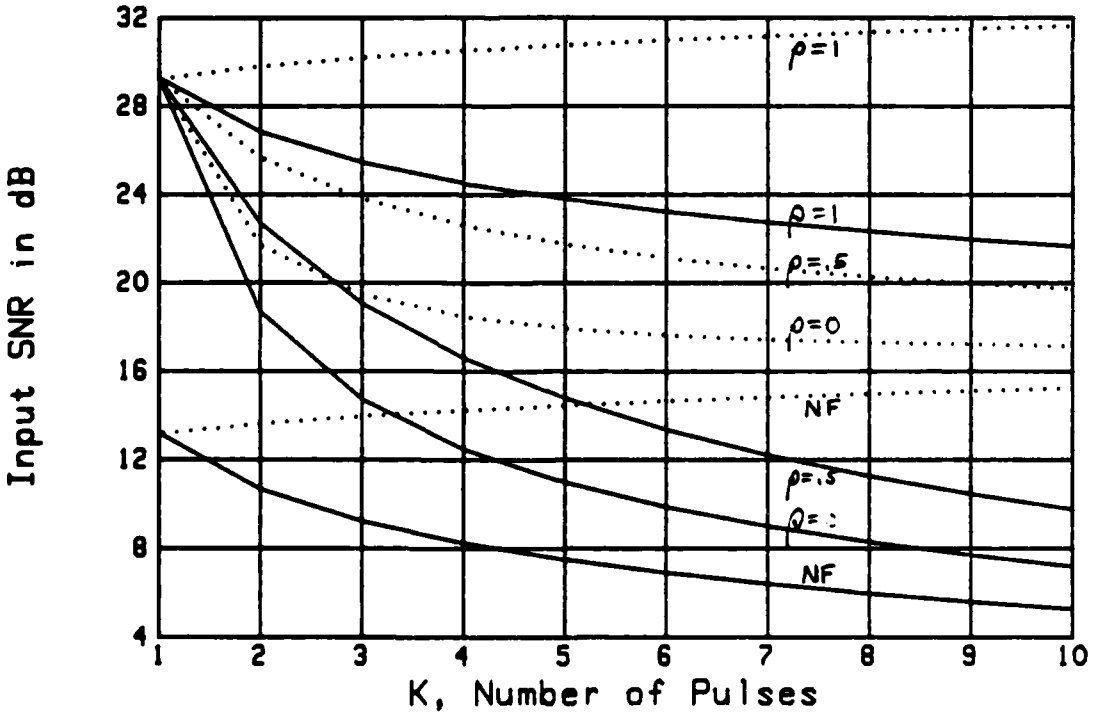



Figure 12. $P_F$ = 1E-6, $P_D$ = .9, m = .5

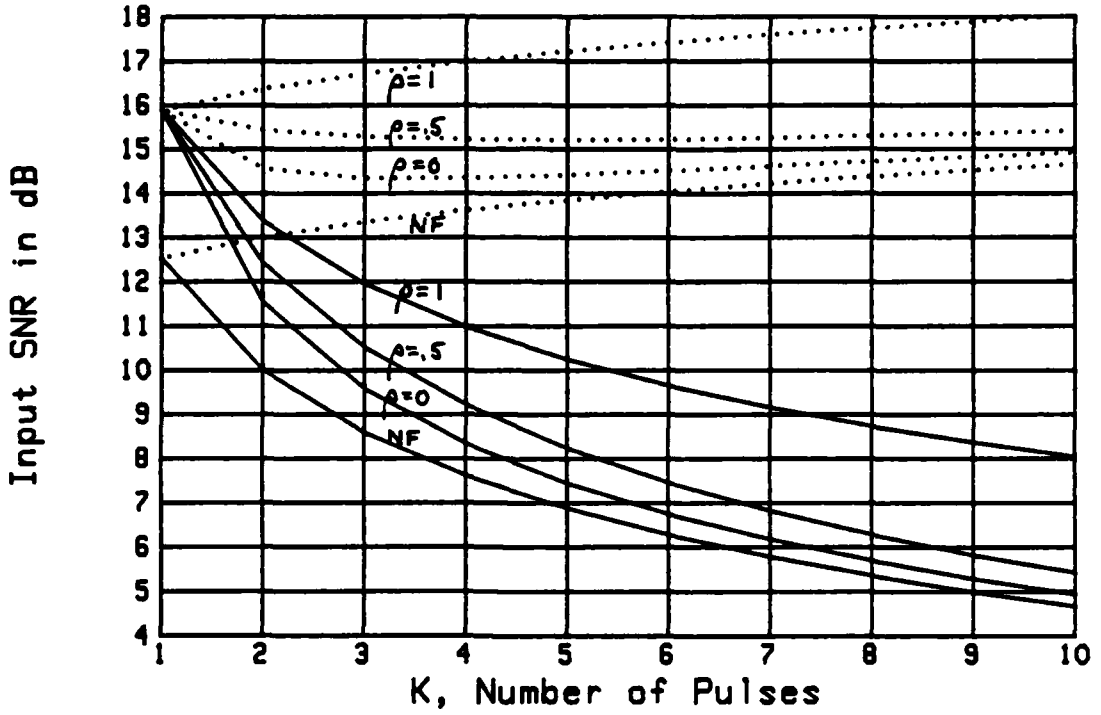



Figure 13. $P_F$ = 1E-8, $P_D$ = .5, m = .5

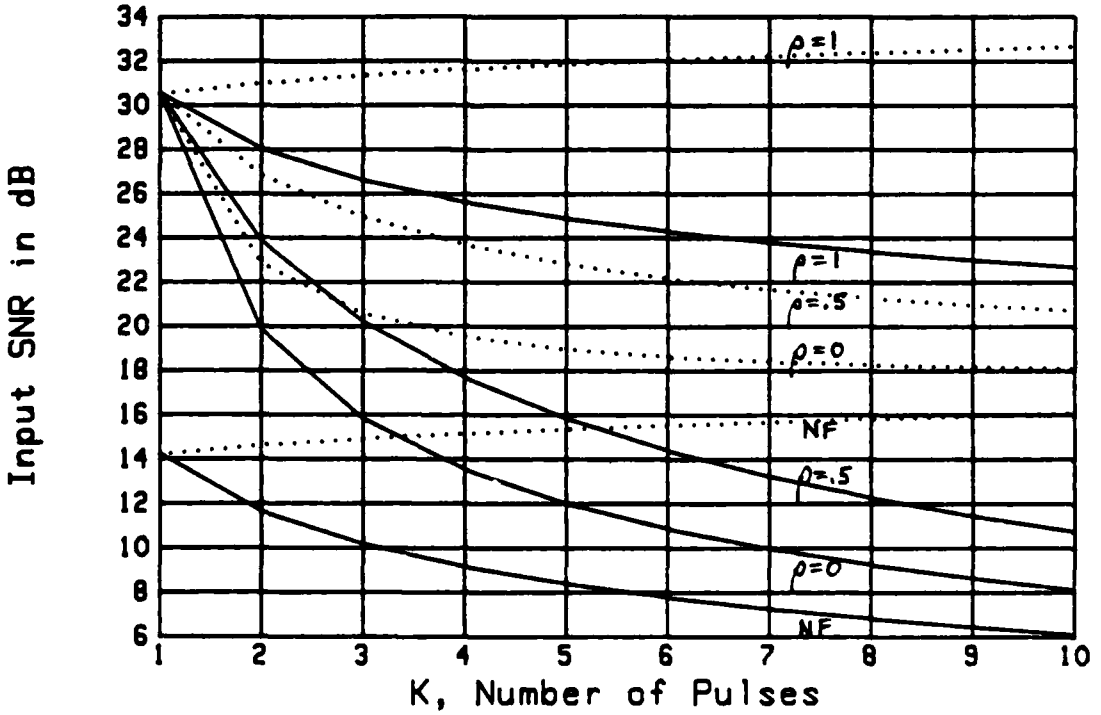



Figure 14. $P_F$ = 1E-8, $P_D$ = .9, m = .5

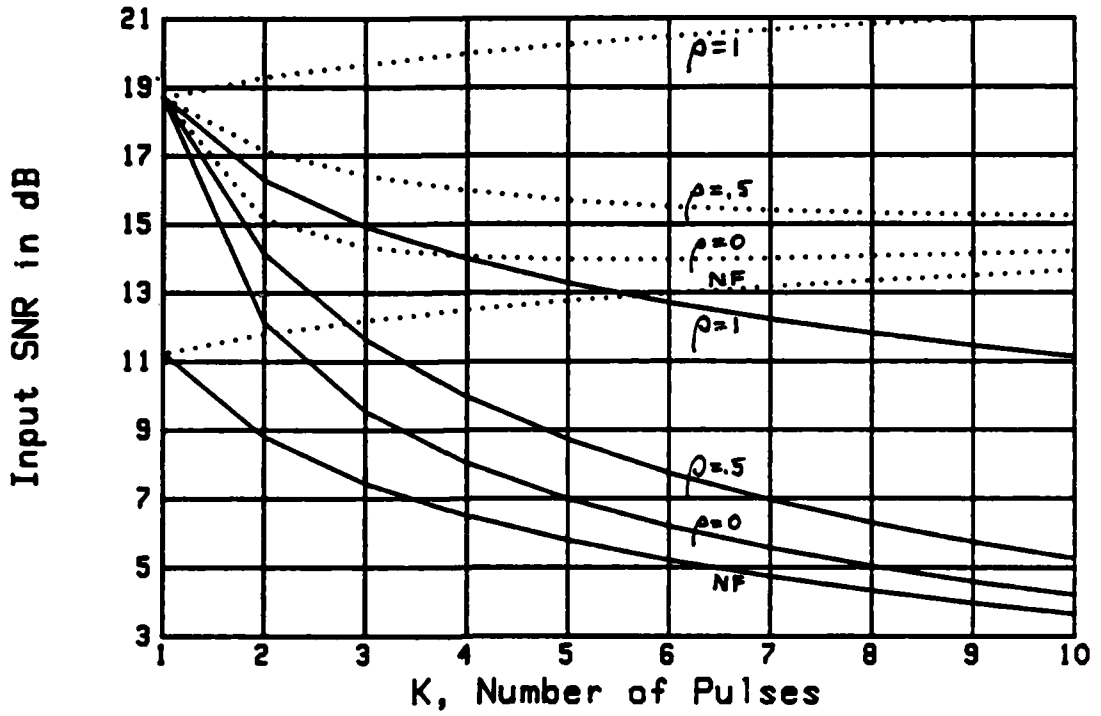



Figure 15. $P_F$ = 1E-6, $P_D$ = .5, m = .25

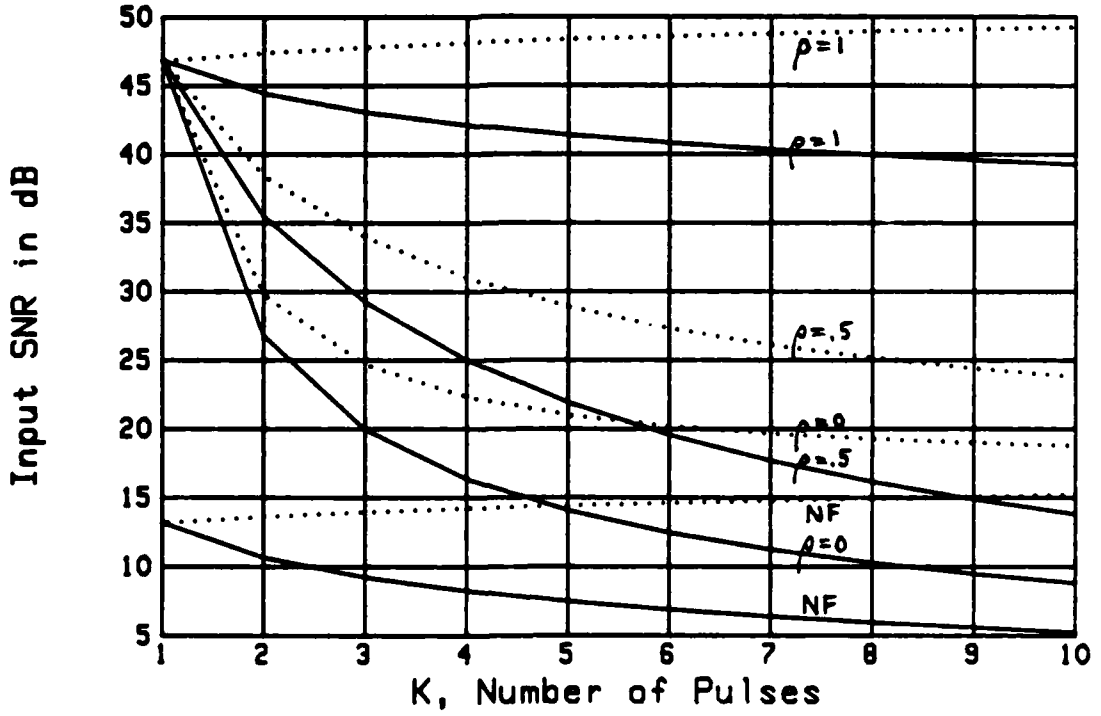



Figure 16. $P_F$ = 1E-6, $P_D$ = .9, m = .25

## SUMMARY

One manner of summarizing the behavior of the amplitude and power scaling variates (figure 2) is to observe that as the parameter m decreases from infinity (the nonfading case), the fading becomes deeper more often. For small m, the very deep fades occur very often. As m nears zero, the fades are so deep and occur so often that large arguments of the scaling probability density functions are rarely observed. Thus, for a given level of performance, the required input signal-to-noise ratio must increase as m decreases.

This conclusion is borne out by the $P_D$ = .9, $P_F$ = 1E-6 case (figures 4, 8, 12, and 16) where, for a single pulse system, K=1, the required input signal-to-noise ratio increases from 17 dB to 47 dB as m decreases from 2 to .25. Relaxing the probability of detection to $P_D$ = .5 relaxes the corresponding signal-to-noise ratios to the range from 12 dB to 19 dB. It is also not surprising that tightening the probability of false alarm to 1E-8 further increases the required signal-to-noise ratio. For m = .25 and $P_F$ = 1E-8, the signal-to-noise ratio requirements are preposterous and the corresponding figures for $P_D$ = .5 and $P_D$ = .9, which would have completed the set, have been omitted.

Although the signal model employed here in (1) and (4) utilizes a common carrier frequency $f_o$ for all K pulses, the results extend immediately to the case of arbitrary different carrier frequencies $f_k$ for the k-th pulse,

provided that these various frequencies are known and appropriately used at the receiver. Also, although the particular signal pulse in (1) is a simple burst of a sinewave, the current results extend to arbitrary signal waveforms $p_k(t)$ on each pulse, provided that matched filters are employed in the receiver, and that each waveform $p_k(t)$ have identical transmitted energy.

# APPENDIX A

## CHARACTERISTIC FUNCTION OF SYSTEM OUTPUT

### Matched Filter Output Characterization

In this appendix, we compute the characteristic function of the decision variable, described in (7) et seq. The in-phase and quadrature outputs of the k-th matched filter, at the time instant of peak signal output, are

$$\begin{Bmatrix} I_k \\ Q_k \end{Bmatrix} = \int_L dt \left[ A\, r_k \cos(2\pi f_o(t - t_k - D) + \theta_k) + n(t) \right] *$$

$$* \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (2\pi f_o(t - t_k - D)) \qquad \text{for } 1 \le k \le K \,, \tag{A-1}$$

where use has been made of (1) and (7), and phase shift $\theta_o$ has been absorbed in random phase shift $\theta_k$, with no loss of generality. That is,

$$I_k = \alpha_k + \mu_k \,, \quad Q_k = \beta_k + \upsilon_k \,, \tag{A-2}$$

where the signal components are, upon use of the facts that $Lf_o \gg 1$ and that there is no overlap, as indicated in (3),

$$\alpha_k = \tfrac{1}{2} AL\, r_k \cos \theta_k \,,$$

$$\beta_k = -\tfrac{1}{2} AL\, r_k \sin \theta_k \,, \tag{A-3}$$

and the noise components are

$$\begin{Bmatrix} \mu_k \\ \upsilon_k \end{Bmatrix} = \int_L dt\; n(t) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (2\pi f_o(t - t_k - D)) \,. \tag{A-4}$$

The noise components in (A-4) are Gaussian zero-mean random variables since

$$\overline{n(t)} = 0 \ , \quad \overline{n(t_1)n(t_2)} = N_d \, \delta(t_1 - t_2) \ . \tag{A-5}$$

There also immediately follows by the use of (A-5),

$$\overline{\mu_k^2} = \overline{\nu_k^2} = N_d \, L \, \frac{1}{2} \equiv \sigma_n^2 \ , \quad \overline{\mu_k \nu_k} = 0 \ . \tag{A-6}$$

Furthermore, due to the lack of overlap, as in (3), the random variables $\mu_k, \nu_k$ are independent of $\mu_j, \nu_j$ for $k \neq j$. Thus, the collection of 2K random variables in (A-4) are all independent and identically distributed.

Conditional Characteristic Function

The k-th squared envelope sample at the matched filter output is

$$\gamma_k = I_k^2 + Q_k^2 = (\alpha_k + \mu_k)^2 + (\beta_k + \nu_k)^2 \ , \tag{A-7}$$

by reference to (A-1) and (A-2). We will temporarily consider that the signal components in (A-3) are fixed non-random constants; then the conditional characteristic function of random variable $\gamma_k$ is, upon use of (A-6),

$$f_k(\xi) = \overline{\exp(i\xi\gamma_k)} = \overline{\exp[i\xi(\alpha_k + \mu_k)^2 + i\xi(\beta_k + \nu_k)^2]} =$$

$$= \iint d\mu \, d\nu \, \frac{1}{2\pi\sigma_n^2} \exp\left(-\frac{\mu^2 + \nu^2}{2\sigma_n^2}\right) \exp[i\xi(\alpha_k + \mu)^2 + i\xi(\beta_k + \nu)^2] \ . \tag{A-8}$$

Now let

$$\mu = \rho \cos \phi \ , \quad \nu = \rho \sin \phi \ , \tag{A-9}$$

to get

$$f_k(\xi) = \int_0^\infty d\rho\ \rho \int_{-\pi}^{\pi} d\phi\ \frac{1}{2\pi\sigma_n^2} \exp\left[-\frac{\rho^2}{2\sigma_n^2} + i\xi\left\{\alpha_k^2 + \beta_k^2 + 2\rho(\alpha_k \cos\phi + \beta_k \sin\phi) + \rho^2\right\}\right] =$$

$$= \exp[i\xi(\alpha_k^2 + \beta_k^2)] \int_0^\infty d\rho\ \frac{\rho}{\sigma_n^2} \exp\left[-\frac{\rho^2}{2\sigma_n^2} + i\xi\rho^2\right] J_0(\xi 2\rho(\alpha_k^2 + \beta_k^2)^{1/2}) =$$

$$= (1 - i\xi 2\sigma_n^2)^{-1} \exp\left[\frac{i\xi(\alpha_k^2 + \beta_k^2)}{1 - i\xi 2\sigma_n^2}\right]. \qquad \text{(A-10)}$$

Here we employed [3; 8.411 1 and 6.631 4].

The output of the system is the sum of the squared envelope samples in (A-7):

$$\gamma = \sum_{k=1}^{K} \gamma_k = \sum_{k=1}^{K} (I_k^2 + Q_k^2)\ . \qquad \text{(A-11)}$$

Since all the noise random variables in (A-4) are independent, the conditional characteristic function of system output random variable $\gamma$ follows from (A-10) as

$$f_\gamma^{(c)}(\xi) = (1 - i\xi 2\sigma_n^2)^{-K} \exp\left[\frac{i\xi}{1 - i\xi 2\sigma_n^2} \sum_{k=1}^{K} (\alpha_k^2 + \beta_k^2)\right]. \qquad \text{(A-12)}$$

If we now use (A-3), this can be expressed as

$$f_\gamma^{(c)}(\xi) = (1 - i\xi 2\sigma_n^2)^{-K} \exp\left[\frac{i\xi}{1 - i\xi 2\sigma_n^2} \left(\frac{AL}{2}\right)^2 \sum_{k=1}^{K} r_k^2\right]. \qquad \text{(A-13)}$$

It is important to observe that this conditional characteristic function depends on the signal amplitude scalings $\{r_k\}$ in (4) and (7) only through the sum of squares

$$S = \sum_{k=1}^{K} r_k^2 = \sum_{k=1}^{K} q_k \ , \qquad \text{(A-14)}$$

where

$$q_k = r_k^2 = \text{power scaling of k-th pulse} \ . \qquad \text{(A-15)}$$

First Order Statistics of Scalings

Let the first order probability density function of amplitude scaling $r_k$ in (4) be

$$p_r(u) = \frac{2\, u^{2m-1} \exp(-u^2/\alpha)}{\Gamma(m)\, \alpha^m} \quad \text{for } u > 0 \quad (m > 0) \ , \qquad \text{(A-16)}$$

where m need not be an integer. (As an example, for m = 1,

$$p_r(u) = \frac{2}{\alpha}\, u \exp(-u^2/\alpha) \quad \text{for } u > 0 \ , \qquad \text{(A-17)}$$

which corresponds to Rayleigh amplitude fading.)

The corresponding probability density function for power scaling $q_k$ in (A-15) is

$$p_q(u) = \frac{u^{m-1} \exp(-u/\alpha)}{\Gamma(m)\, \alpha^m} \quad \text{for } u > 0 \quad (m > 0) \ . \qquad \text{(A-18)}$$

This is recognized as the probability density function of a chi-squared random variate with 2m degrees of freedom; however, 2m need not be an integer here. The general $v$-th moment of $q_k$ is

$$\overline{q_k^v} = \alpha^v \frac{\Gamma(m + v)}{\Gamma(m)} , \tag{A-19}$$

and, in particular, we have

$$\bar{q} = \alpha m, \quad \text{Var}(q) = \alpha^2 m, \quad \frac{\text{Std Dev}(q)}{\bar{q}} = \frac{1}{\sqrt{m}} . \tag{A-20}$$

The first order characteristic function of power scaling $q_k$ is given by the Fourier transform of (A-18):

$$f_q(\xi) = \int du \, \exp(i\xi u) \, p_q(u) = (1 - i\xi\alpha)^{-m} . \tag{A-21}$$

First Order Characteristic Function of Sum S in Special Cases

If all the power scalings $\{q_k\}$ in sum S in (A-14) were independent, then the characteristic function of S would be given by

$$(1 - i\xi\alpha)^{-mK} , \tag{A-22}$$

as may be seen from (A-21). This situation could come about approximately when the signal pulses in (2), (4), and (7) are widely separated in time and subject to uncorrelated (linearly independent) fading.

On the other hand, if all the power scalings were completely dependent, then the characteristic function of sum S would be

$$(1 - i\xi\alpha K)^{-m} \qquad \text{(A-23)}$$

since $S = Kq_1$ in this case; see (A-14) and (A-21). This case corresponds to close signal pulses and/or very slow fading.

Approximate First Order Characteristic Function of Sum S

The latter two results suggest the following form* as an approximation to the characteristic function of sum S in (A-14) when the power scalings $\{q_k\}$ are partially dependent:

$$f_S(\xi) \tilde{=} (1 - i\xi\beta)^{-mK_e} . \qquad \text{(A-24)}$$

This latter form of characteristic function has mean $mK_e\beta$ and variance $mK_e\beta^2$. We will choose the two unknown parameters in (A-24) so that these two statistics identically equal the corresponding exact values determined directly from (A-14). Specifically, using (A-20), mean

$$\bar{S} = K\,\bar{q} = K\,\alpha\,m , \qquad \text{(A-25)}$$

---

*A similar procedure was employed with great success for a spectral analysis technique in [4; (38) et seq.].

and variance

$$\mathrm{Var}(S) = \overline{(S - \bar{S})^2} = \overline{\left[\sum_{k=1}^{K} (q_k - \bar{q})\right]^2} =$$

$$= \sigma_q^2 \sum_{k,\ell=1}^{K} \rho_{k\ell} = \alpha^2 m \sum_{k,\ell=1}^{K} \rho_{k\ell} \ , \qquad \text{(A-26)}$$

where

$$\rho_{k\ell} = \frac{1}{\sigma_q^2} \overline{(q_k - \bar{q})(q_\ell - \bar{q})} \qquad \text{(A-27)}$$

is the normalized covariance coefficient of power scalings $\{q_k\}$. Combining these quantities, we find that

$$K_e = \frac{K^2}{\sum_{k,\ell=1}^{K} \rho_{k\ell}} \ ,$$

$$\beta = \alpha \frac{K}{K_e} = \frac{\bar{q}}{m} \frac{K}{K_e} \ . \qquad \text{(A-28)}$$

The quantity $K_e$ in (A-24) and (A-28) can be interpreted as an effective number of independent scalings in (A-14). For example, if $\rho_{k\ell} = 0$ for $k \neq \ell$, then $K_e = K$; while if $\rho_{k\ell} = 1$ for all $k, \ell$, then $K_e = 1$. Both of these situations agree with physical intuition. Also, from (A-28), $\beta = \alpha$ in the former case, and $\beta = \alpha K$ in the latter case; (A-24) then reduces to (A-22) and (A-23), respectively, as required.

No specific time separations $\{t_k\}$, in (4) and (7), need be assumed for (A-28) to apply. Some of the pulses can be close together, while others can be widely separated, subject of course to limitation (3). The only way that the statistical dependencies of the power scalings $\{q_k\}$ in sum S enter the characteristic function of S is through the double summation of covariance coefficients in (A-28).

Unconditional Characteristic Function of Output $\gamma$

The first order probability density function corresponding to characteristic function (A-24) is

$$p_S(u) = \frac{u^{N-1} \exp(-u/\beta)}{\Gamma(N)\, \beta^N} \quad \text{for } u > 0 , \tag{A-29}$$

where

$$N = m K_e . \tag{A-30}$$

None of the parameters, N, m, $K_e$, need be integer. Also, N can be larger or smaller than K, the number of signal pulses.

The conditional characteristic function of system output $\gamma$ was given in (A-13). The unconditional characteristic function of $\gamma$ is obtained by averaging (A-13) with respect to (A-29):

$$f_\gamma(\xi) = \int_0^\infty du \, \frac{u^{N-1} \exp(-u/\beta)}{\Gamma(N)\, \beta^N} (1 - i\xi 2\sigma_n^2)^{-K} \exp\left[\frac{i\xi}{1 - i\xi 2\sigma_n^2}\left(\frac{AL}{2}\right)^2 u\right] =$$

$$= (1 - i\xi 2\sigma_n^2)^{N-K} (1 - i\xi 2\sigma_T^2)^{-N} , \qquad \text{(A-31)}$$

where

$$\sigma_T^2 = \sigma_n^2\left(1 + \frac{\overline{E_1}}{N_o}\frac{K}{N}\right). \qquad \text{(A-32)}$$

To obtain this last relation, we used (A-6), (A-28), (A-30), (6), and

$$N_o = 2N_d , \qquad \text{(A-33)}$$

the single-sided noise density level stated just above (7).

Equation (A-31) is a compact simple expression for the characteristic function of decision variable $\gamma$. It could be used directly in the efficient procedures of [5] to get the cumulative and/or exceedance distributions of $\gamma$. In the next appendix, we derive analytic expressions for these probabilities.

# APPENDIX B

## EXCEEDANCE DISTRIBUTION OF SYSTEM OUTPUT

The characteristic function of decision variable $\gamma$ is given in (A-31). In order to find the corresponding exceedance distribution function, we use the procedure in [6; appendix A]: since we know the characteristic function-probability density function pair

$$\frac{1}{(1 - i\$a)^J} \leftrightarrow \frac{u^{J-1} \exp(-u/a)}{\Gamma(J)\, a^J} \quad \text{for } u > 0, \ (J > 0) , \tag{B-1}$$

the probability density function corresponding to multiplicative characteristic function (A-31) is given by convolution

$$p_\gamma(u) = \int_0^u dx \, \frac{x^{J-1} \exp(-x/a)}{\Gamma(J)\, a^J} \, \frac{(u - x)^{N-1} \exp(-(u - x)/b)}{\Gamma(N)\, b^N} \quad \text{for } u > 0 , \tag{B-2}$$

where

$$J = K - N, \quad a = 2\sigma_n^2, \quad b = 2\sigma_T^2 . \tag{B-3}$$

We presume $N < K$ for now, in order that $J > 0$.

Employing [3; 3.383 1 and 8.384 1], (B-2) becomes

$$p_\gamma(u) = \frac{\exp(-u/a)\, u^{K-1}}{a^{K-N}\, b^N\, \Gamma(K)} \; {}_1F_1\left(N;\, K;\, u\left(\frac{1}{a} - \frac{1}{b}\right)\right) \quad \text{for } u > 0. \tag{B-4}$$

Since (A-31) is an analytic function of N, as is its transform (B-4), we can now analytically continue (B-4) to values of $N \geq K$.

The exceedance distribution function of $\gamma$ is

$$\text{Prob}(\gamma > u) = Q_\gamma(u) = \int_u^\infty dx\, p_\gamma(x) =$$

$$= \frac{1}{\Gamma(K)\,(b/a)^N} \int_{u/a}^\infty dt\, e^{-t}\, t^{K-1}\, {}_1F_1\left(N;\, K;\, \frac{b-a}{b}\, t\right) . \qquad \text{(B-5)}$$

Reference to (B-3) and (A-32) yields

$$Q_\gamma(u) = \frac{1}{\Gamma(K)\,(1+R)^N} \int_\Lambda^\infty dt\, e^{-t}\, t^{K-1}\, {}_1F_1\left(N;\, K;\, \frac{R}{1+R}\, t\right) , \qquad \text{(B-6)}$$

where

$$R = \frac{\overline{E_1}}{N_o}\,\frac{K}{N} , \quad \Lambda = \frac{u}{2\sigma_n^2} . \qquad \text{(B-7)}$$

By expanding ${}_1F_1$ in a power series according to [3; 9.210 1], integrating term by term, and using the partial exponential [1; 6.5.11]

$$e_j(x) = \sum_{n=0}^{j} x^n/n! , \qquad \text{(B-8)}$$

(B-6) develops into the form

$$Q_\gamma(u) = \frac{\exp(-\Lambda)}{(1+R)^N} \sum_{n=0}^{\infty} \frac{(N)_n}{n!} \left(\frac{R}{1+R}\right)^n e_{K-1+n}(\Lambda) . \qquad \text{(B-9)}$$

This series for the exceedance distribution function converges for all R, but rather slowly for large R.

For signal present, (B-9) gives the detection probability. When signal is absent, R = 0 from (B-7), and (B-9) reduces to the false alarm probability

$$P_F = \exp(-\lambda)\, e_{K-1}(\lambda) \ , \tag{B-10}$$

which is independent of N (defined in (A-30) and (A-28)).

An alternative form to (B-9) is available if we employ Kummer's transformation [1; 13.1.27] in (B-6):

$$Q_\gamma(u) = \exp\left(\frac{-\lambda}{1+R}\right)(1+R)^{K-N} \sum_{n=0}^{\infty} \frac{(K-N)_n}{n!}(-R)^n\, e_{K-1+n}\left(\frac{\lambda}{1+R}\right). \tag{B-11}$$

If N is an integer and if $N \geq K$, this series terminates, and is composed of all positive terms, which makes it very attractive. However, if either condition is violated, then (B-11) converges only for $R < 1$, but diverges for $R > 1$. Since this latter case is of practical significance, (B-11) is then not too useful for general values of N.

The only convergence factor in (B-9) is the $R/(1+R)$ term. However, we can create another convergence term by adding and subtracting $\exp(\lambda)$ from the partial exponential, since this is its limit value as n tends to infinity; see (B-8). By then employing the result that [3; 9.100 and 9.121 1]

$$\sum_{n=0}^{\infty} \frac{(N)_n}{n!}\left(\frac{R}{1+R}\right)^n = {}_2F_1\left(N, \beta; \beta; \frac{R}{1+R}\right) = (1+R)^N \ , \tag{B-12}$$

(B-9) becomes

$$Q_\gamma(u) = 1 - \frac{\exp(-\Lambda)}{(1+R)^N} \sum_{n=0}^{\infty} \frac{(N)_n}{n!} \left(\frac{R}{1+R}\right)^n \left[\exp(\Lambda) - e_{K-1+n}(\Lambda)\right] . \qquad (B\text{-}13)$$

This last form was used for all the numerical results obtained here; a program for this expansion is given in appendix C. If necessary, a more elaborate and accurate procedure involving only positive summations is developed in [6; pages A-6 to A-8].

To summarize, the parameters required in detection probability (B-13) are

$$\Lambda = \frac{u}{2\sigma_n^2} , \quad R = \frac{\overline{E_1}}{N_o} \frac{K}{N} , \qquad (B\text{-}14)$$

where $\sigma_n^2$ is given by (A-6) and N is given by (A-30) and (A-28). Since $\Lambda$ is simply a scaled version of u, we can equally well keep $\Lambda$ as the fundamental threshold variable in (B-10) and (B-13).

# APPENDIX C

## PROGRAM LISTINGS

Two programs are listed here, the first for fading, the second for no-fading, of the received signal. Inputs required of the user for the fading case are given in lines 20 - 50:

| | |
|---|---|
| Pf | probability of false alarm |
| Pd | probability of detection |
| M | parameter m in (11) and (13) |
| Rho | $\rho_{k\ell} = \rho^{\lvert k-\ell \rvert}$ in (10) and (15) . |

The particular exponential normalized covariance $\rho_{k\ell}$ programmed in line 1490 can be easily replaced by other more general cases of interest to the user.

The output of the main program, line 290, is the value in dB of $\overline{E}_1/N_o$ in (9), namely a measure of the required system input signal-to-noise ratio to achieve the desired false alarm and detection probabilities. The number of pulses, K, is taken to be 1 to 10 in line 120, but can be easily changed. The first inverse function subroutine in line 770 solves for the required threshold for specified $P_F$ and K. The second subroutine in line 1130 solves for the required $\overline{E}_1/N_o$ for specified $P_D$ and N.

A similar explanation holds for the no-fading program listed at the end of the appendix. Duplicate routines are not listed in full, in order to save space. The word DOUBLE denotes INTEGER variables in Hewlett Packard BASIC on the 9000 computer.

```
 10  ! SNR-FADING
 20    Pf=1.E-6                  !  FALSE ALARM PROBABILITY
 30    Pd=.5                     !  DETECTION PROBABILITY
 40    M=1.                      !  FADING PARAMETER (SINGLE PULSE)
 50    Rho=.5                    !  FADING CORRELATION COEFFICIENT
 60    A$="1E-6,.5,1.,.5"        !  Pf,Pd,M,Rho
 70    Ef=1.E-20                 !  TOLERANCE ON Pf
 80    Ed=1.E-15                 !  TOLERANCE ON Pd
 90    DIM E1nodb(1:10)          !  REQUIRED INPUT SNR PER PULSE (DB)
100    COM DOUBLE K,REAL N,Thr
110    DOUBLE Ks,Ls
120    FOR K=1 TO 10             !  NUMBER OF PULSES
130    S=0.
140    FOR Ks=1 TO K
150    FOR Ls=1 TO K
160    S=S+FNCovnorm(Ks,Ls,Rho)
170    NEXT Ls
180    NEXT Ks
190    Keff=K*K/S
200    N=M*Keff
210    Thrinc=1.                 !  THRESHOLD INCREMENT
 20    Thrstart=-LOG(Pf)-Thrinc  !  THRESHOLD STARTING VALUE
230    CALL Inversfunction1(-Pf,Ef,Thrstart,Thrinc,Thr)
240    E1nostart=10.             !  E1/No STARTING VALUE
250    E1noinc=1.                !  E1/No INCREMENT
260    CALL Inversfunction2(Pd,Ed,E1nostart,E1noinc,E1no)
270    E1nodb(K)=10.*LGT(E1no)
280    NEXT K
290    PRINT E1nodb(*)
300    CREATE DATA A$,4
310    ASSIGN #1 TO A$
320    PRINT #1;E1nodb(*)
330    ASSIGN #1 TO *
340    END
350    !
360    DEF FNPf(Thr)             !  FALSE ALARM PROBABILITY
370    COM DOUBLE K
380    DOUBLE Ks
390    S=T=EXP(-Thr)
400    FOR Ks=1 TO K-1
410    T=T*Thr/Ks
420    S=S+T
430    NEXT Ks
440    RETURN -S                 !  - TO YIELD INCREASING FUNCTION
450    FNEND
460    !
```

```
470     DEF FNPd(Eino)          !  FADING DETECTION PROBABILITY
480     COM DOUBLE K,REAL N,Thr
490     Error=1.E-10
500     DOUBLE K1,Ks
510     R=Eino*K/N
520     Et=EXP(Thr)
530     K1=K-1
540     N1=N-1.
550     R1=1.+R
560     Q=R/R1
570     E=Te=1.
580     FOR Ks=1 TO K1
590     Te=Te*Thr/Ks
600     E=E+Te
610     NEXT Ks
620     S=B=MAX(Et-E,0.)
630     T=1.
640     FOR Ks=1 TO 1000
650     Te=Te*Thr/(K1+Ks)
660     B=MAX(B-Te,0.)
670     T=T*Q*(N1+Ks)/Ks
680     Pr=T*B
690     S=S+Pr
700     IF ABS(Pr)<=Error*ABS(S) THEN 730
710     NEXT Ks
720     PRINT "1000 TERMS AT:";K;N;Thr;Eino;Pr/S
730     Pd=1.-EXP(-Thr-N*LOG(R1))*S
740     RETURN Pd
750     FNEND
760     !
770     SUB Inversfunction1(Desired,Error,X1,Del,X2)
780     X2=X1+Del
790     F1=FNPf(X1)
800     F2=FNPf(X2)
810     IF F2>=Desired THEN 860
820     X1=X2
830     X2=X2+Del
840     F1=F2
850     GOTO 800
860     IF F1<=Desired THEN 920
870     X2=X1
880     X1=X1-Del
890     F2=F1
900     F1=FNPf(X1)
910     GOTO 860
920     Xa=X1
930     Xb=X2
940     IF F2-Desired<Desired-F1 THEN 1010
950     T=X1
960     X1=X2
970     X2=T
980     T=F1
990     F1=F2
1000    F2=T
```

```
1010    IF ABS(F2-Desired)<Error THEN 1110
1020    IF F2=F1 THEN 1110
1030    T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
1040    T=MAX(T,Xa)
1050    T=MIN(T,Xb)
1060    X1=X2
1070    X2=T
1080    F1=F2
1090    F2=FNPf(X2)
1100    GOTO 1010
1110    SUBEND
1120    !
1130    SUB Inversfunction2(Desired,Error,X1,Del,X2)
1140    X2=X1+Del
1150    F1=FNPd(X1)
1160    F2=FNPd(X2)
1170    IF F2>=Desired THEN 1220
1180    X1=X2
1190    X2=X2+Del
1200    F1=F2
1210    GOTO 1160
1220    IF F1<=Desired THEN 1280
1230    X2=X1
1240    X1=X1-Del
1250    F2=F1
1260    F1=FNPd(X1)
1270    GOTO 1220
1280    Xa=X1
1290    Xb=X2
1300    IF F2-Desired<Desired-F1 THEN 1370
1310    T=X1
1320    X1=X2
1330    X2=T
1340    T=F1
1350    F1=F2
1360    F2=T
1370    IF ABS(F2-Desired)<Error THEN 1470
1380    IF F2=F1 THEN 1470
1390    T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
1400    T=MAX(T,Xa)
1410    T=MIN(T,Xb)
1420    X1=X2
1430    X2=T
1440    F1=F2
1450    F2=FNPd(X2)
1460    GOTO 1370
1470    SUBEND
1480    !
1490    DEF FNCovnorm(DOUBLE Ks,Ls,REAL Rho)
1500    RETURN Rho*ABS(Ks-Ls)
1510    FNEND
```

```
 10  ! SNR-NO-FADING
 20    Pf=1.E-6                  !  FALSE ALARM PROBABILITY
 30    Pd=.5                     !  DETECTION PROBABILITY
 40    A$="1E-6,.5"              !  Pf,Pd
 50    Ef=1.E-20                 !  TOLERANCE ON Pf
 60    Ed=1.E-15                 !  TOLERANCE ON Pd
 70    DIM E1nodb(1:10)          !  REQUIRED INPUT SNR PER PULSE (DB)
 80    COM DOUBLE K,REAL Thr2
 90    FOR K=1 TO 10             !  NUMBER OF PULSES
100    Thrinc=1.                 !  THRESHOLD INCREMENT
110    Thrstart=-LOG(Pf)-Thrinc  !  THRESHOLD STARTING VALUE
120    CALL Inversfunction1(-Pf,Ef,Thrstart,Thrinc,Thr)
130    Thr2=SQR(2.*Thr)
140    E1nostart=10.             !  E1/No STARTING VALUE
150    E1noinc=1.                !  E1/No INCREMENT
160    CALL Inversfunction2(Pd,Ed,E1nostart,E1noinc,E1no)
170    E1nodb(K)=10.*LGT(E1no)
180    NEXT K
190    PRINT E1nodb(*)
200    CREATE DATA A$,4
210    ASSIGN #1 TO A$
220    PRINT #1;E1nodb(*)
230    ASSIGN #1 TO *
240    END
250    !
260    DEF FNPf(Thr)             !  FALSE ALARM PROBABILITY
270    COM DOUBLE K
280    DOUBLE Ks
290    S=T=EXP(-Thr)
300    FOR Ks=1 TO K-1
310    T=T*Thr/Ks
320    S=S+T
330    NEXT Ks
340    RETURN -S                 !  - TO YIELD INCREASING FUNCTION
350    FNEND
360    !
370    DEF FNPd(E1no)            !  NO-FADING DETECTION PROBABILITY
380    COM DOUBLE K,REAL Thr2
390    Pd=FNQm(K,SQR(2.*K*E1no),Thr2)
400    RETURN Pd
410    FNEND
420    !
```

```
430     DEF FNQm(DOUBLE M,REAL A,B)     !   QM(A,B)
440     Error=1.E-17
450     DOUBLE M1,J
460     Q3=.5*A*A
470     Q4=.5*B*B
480     Q5=EXP(-.5*(Q3+Q4))
490     Q6=Q7=Q5
500     M1=M-1
510     FOR J=1 TO M1
520     Q7=Q7*Q4/J
530     Q6=Q6+Q7
540     NEXT J
550     Qm=Q5*Q6
560     FOR J=1 TO 1000
570     Q5=Q5*Q3/J
580     Q7=Q7*Q4/(J+M1)
590     Q6=Q6+Q7
600     Q9=Q5*Q6
610     Qm=Qm+Q9
620     IF Q9<=Error*Qm THEN 650
630     NEXT J
640     PRINT "1000 TERMS IN FNQm(M,A,B) AT ";M;A;B
650     RETURN Qm
660     FNEND
670     !
        SUB Inversfunction1(Desired,Error,X1,Del,X2)
        !  LISTED ABOVE

        SUB Inversfunction2(Desired,Error,X1,Del,X2)
        !  LISTED ABOVE
```

# REFERENCES

1. Handbook of Mathematical Functions, U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series No. 55, U. S. Government Printing Office, June 1964.

2. A. H. Nuttall, Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly from Characteristic Functions, NUSC Technical Report 7023, 1 October 1983.

3. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, Inc., New York, 1980.

4. A. H. Nuttall, Probability Distribution of Spectral Estimates Obtained via Overlapped FFT Processing of Windowed Data, NUSC Technical Report 5529, 3 December 1976.

5. A. H. Nuttall, Signal Processing Studies, NUSC Scientific and Engineering Studies, New London, CT, 1985.

6. A. H. Nuttall, Operating Characteristics for Detection of a Fading Signal with K Dependent Fades and D-fold Diversity in M Alternative Locations, NUSC Technical Report 5739, 25 October 1977.

# INIITAL DISTRIBUTION LIST

| Addressee | No. of Copies |
|---|---|
| ASN (RE&S) | 1 |
| OUSDR&E (Research and Advanced Technology) | 2 |
| DEPUTY USDR&E (Res & Adv Tech) | 1 |
| DEPUTY USDR&E (Dir Elect & Phys Sc) | 1 |
| ONR, ONR-100, -102, -200, -400, -410, -422, -425AC, -430 | 8 |
| COMSPAWARSYSCOM, SPAWAR 05 (W. R. Hunt) | 1 |
| CNO, OP-098, OP-941, OP-951, Code 411 (N. Gerr) | 4 |
| DIA (DT-2C) | 10 |
| NRL, Code 5132 (Dr. P. B. Abraham) Code 5370 (W. Gabriel), Code 5135 (N. Yen) (A. A. Gerlach) | 4 |
| USRD | 1 |
| NORDA | 1 |
| USOC, Code 240, Code 241 | 2 |
| NAVSUBSUPACNLON | 1 |
| NAVOCEANO, Code 02 | 2 |
| NAVELECSYSCOM, ELEX 03, 310 | 2 |
| NAVSEASYSCOM, SEA-00, -05R, -06F, 63D (E. L. Plummer, CDR E. Graham IV), 63R (C. C. Walker), -92R | 8 |
| NAVAIRDEVCEN, Warminster | 1 |
| NAVAIRDEVCEN, Key West | 1 |
| NOSC, Code 8302, Code 6565 (Library), Code 713 (F. Harris) | 3 |
| NAVWPNSCEN | 1 |
| NCSC, Code 724 | 1 |
| NAVCIVENGRLAB | 1 |
| NAVSWC | 1 |
| NAVSURFWPNCEN, Code U31 | 1 |
| NISC | 1 |
| CNET, Code 017 | 1 |
| CNTT | 1 |
| NAVSUBSCOL | 1 |
| NAVTRAEQUIPCENT, Technical Library | 1 |
| NAVPGSCOL | 2 |
| NAVWARCOL | 1 |
| NETC | 1 |
| APL/UW, SEATTLE | 1 |
| ARL/PENN STATE, STATE COLLEGE | 1 |
| CENTER FOR NAVAL ANALYSES (ACQUISITION UNIT) | 1 |
| DTIC | 2 |
| DARPA, Alan Ellinthorpe | 1 |
| NOAA/ERL | 1 |
| NATIONAL RESEARCH COUNCIL | 1 |
| WOODS HOLE OCEANOGRAPHIC INSTITUTION | 1 |
| ENGINEERING SOCIETIES LIB, UNITED ENGRG CTR | 1 |
| NATIONAL INSTITUTE OF HEALTH | 1 |
| ARL, UNIV OF TEXAS | 1 |

# INIITAL DISTRIBUTION LIST

| Addressee | No. of Copies |
|---|---|
| MARINE PHYSICAL LAB, SCRIPPS | 1 |
| UNIVERSITY OF CALIFORNIA, SAN DIEGO | 1 |
| NAVSURWEACTR | 1 |
| DELSI | 1 |
| DIRECTOR SACLANT ASW RES CEN | 1 |
| COM SPACE & NAV WAR SYS COM | 1 |
| COM NAVAL PERSONNEL R&D CENTER | 1 |
| COM NAV SUB COLLEGE | 1 |
| B-K DYN INC | 1 |
| BBN, Arlington, VA (Dr. H. Cox) | 1 |
| BBN, Cambridge, MA (H. Gish) | 1 |
| BBN, New London, CT (Dr. P. Cable) | 1 |
| EWASCTRI | 1 |
| MAR, INC, East Lyme, CT | 1 |
| HYDROINC (D. Clark) | 1 |
| SUMRESCR (M. Henry) | 1 |
| ANALTECHNS, N. Stonington, CT | 1 |
| ANALTECHNS, New London, CT | 1 |
| EDOCORP (J. Vincenzo) | 1 |
| TRA CORP., Austin, TX (Dr. T. Leih, J. Wilkinson) | 2 |
| TRA CORP., Groton, CT | 1 |
| NETS (R. Medeiros) | 1 |
| GESY, D. Bates | 1 |
| SONALYSTS, Waterford, CT (J. Morris) | 1 |
| ORI CO, INC. (G. Assard) | 1 |
| HUGHES AIRCRAFT CO. (S. Autrey) | 1 |
| MIT (Prof. A. Baggaroer) | 1 |
| RAYTHEON CO. (J. Bartram) | 1 |
| Dr. Julius Bendat, 833 Moraga Dr, Los Angeles, CA | 1 |
| COOLEY LABORATORY (Prof. T. Birdsall) | 1 |
| PROMETHEUS, INC. (Dr. James S. Byrnes) | 1 |
| BBN INC., New London, CT (Dr. P. Cable) | 1 |
| BBN INC., Arlington VA (Dr. H. Cox) | 1 |
| BBN INC., Cambridge, MA (H. Gish) | 1 |
| ROYAL MILITARY COLLEGE OF CANADA (Prof. Y. T. Chan) | 1 |
| UNIV. OF FLORIDA (D. C. Childers) | 1 |
| SANDIA NATIONAL LABORATORY (J. Claasen) | 1 |
| COGENT SYSTEMS, INC. (J. P. Costas) | 1 |
| IBM CORP. (G. Demuth) | 1 |
| UNIV. OF STRATHCLYDE, CLASGOW, SCOTLAND (Prof. T. Durrani) | 1 |
| ROCKWELL INTERNATIONAL CORP. (L. T. Einstein and Dr. D. F. Elliott) | 2 |
| GENERAL ELECTRIC CO. (Dr. M. Fitelson) | 1 |
| HONEYWELL, INC. (D. M. Goodfellow, Dr. Murray Simon, W. Hughey) | 3 |
| UNIV. OF TECHNOLOGY, LOUGHBOROUGH, LEICESTERSHIRE, ENGLAND (Prof. J. W. R. Griffiths) | 1 |
| HARRIS SCIENTIFIC SERVICES (B. Harris) | 1 |

# INITIAL DISTRIBUTION LIST

| Addressee | No. of Copies |
|---|---|
| UNIV OF CALIFORNIA, SAN DIEGO (Prof. C. W. Helstrom) | 1 |
| EG&G (Dr. J. Hughen) | 1 |
| A&T, INC. (H. Jarvis) | 1 |
| BELL COMMUNICATIONS RESEARCH (J. F. Kaiser) | 1 |
| UNIV. OF RI (Prof. S. Kay, Prof. L. Scharf, Prof. D. Tufts) | 3 |
| MAGNAVOX GOV. & IND. ELEC. CO. (R. Kenefic) | 1 |
| DREXEL UNIV. (Prof. Stanislav Kesler) | 1 |
| UNIV. OF CT (Prof. C. H. Knapp) | 1 |
| APPLIED SEISMIC GROUP (R. Lacoss) | 1 |
| ADMIRALITY RESEARCH ESTABLISHMENT, ENGLAND (Dr. L. J. Lloyd) | 1 |
| NAVAL SYSTEMS DIV., SIMRAD SUBSEA A/S, NORWAY (E. B. Lunde) | 1 |
| DEFENCE RESEARCH ESTAB. ATLANTIC, DARTMOUTH, NOVA SCOTIA CANADA (B. E. Mackey, Library) | 1 |
| MARTIN MARIETTA AEROSPACE (S. L. Marple) | 1 |
| PSI MARINE SCIENCES (Dr. R. Mellen) | 1 |
| Dr. D. Middleton, 127 E. 91st St. NY, NY 10128 | 1 |
| Dr. P. Mikhalevsky, 803 W. Broad St. Falls Church, VA 22046 | 1 |
| CANBERRA COLLEGE OF ADV. EDUC., AUSTRALIA 2616 (P. Morgan) | 1 |
| NORTHEASTERN UNIV., (Prof. C. L. Nikias) | 1 |
| ASTRON RESEARCH & ENGR, (Dr. A. G. Piersol) | 1 |
| WESTINGHOUSE ELEC. CORP. (Dr. H. L. Price) | 1 |
| M/A-COM GOVT SYSTEMS, (Dr. R. Price) | 1 |
| DALHOUSIE UNIV. HALIFAX, NOVA SCOTIA (Dr. B. Ruddick) | 1 |
| NATO SACLANT ASW RESEARCH CNTER, LIBRARY, APO NY, NY 09019 | 1 |
| YALE UNIV. (Prof. M. Schultheiss) | 1 |
| NATIONAL RADIO ASTRONOMY OBSERVATORY (F. Schwab) | 1 |
| DEFENSE SYSTEMS, INC (Dr. G. S. Sebestyen) | 1 |
| DEPT OF OCEAN ENGR. (Dr. R. C. Spindel, Dr. E. Weinstein) | 2 |
| NATO SACLANT ASW RESEARCH CENTRE (Dr. E. J. Sullivan) | 1 |
| PENN STATE UNIV, APPLIED RESEARCH LAB. (F. W. Symons) | 1 |
| NAVAL PG SCHOOL, (Prof. C.W. Therrien) | 1 |
| DEFENCE RESEARCH ESTABLISHMENT PACIFIC, VICTORIA, B.C. CANADA VOS 1BO (Dr. D. J. Thomson) | 1 |
| Robert J. Urick, 11701 Berwick Rd, Silver Spring, MD 20904 | 1 |
| RCA CORP (H. Urkowitz) | 1 |
| USEA S.P.A. LA SPEZIA, ITALY (H. Van Asselt) | 1 |
| NORDA, Code 345 (R. Wagstaff) | 1 |
| TEL-AVIV UNIV. ISRAEL (Prof. E. Weinstein) | 1 |
| COAST GUARD ACADEMY (Prof. J. J. Wolcin) | 1 |
| SPACE PHYSICS LAB, UNIV OF ALBERTA, EDMONTON, CANADA (K. L. Yeung) | 1 |
| UNIV. OF IOWA (Prof. D. H. Youn) | 1 |

END

10-86

DTIC